Locally-Measured Rényi Divergences

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***Abstract*—We propose an extension of the classical Rényi divergences to quantum states through an optimization over probability distributions induced by restricted sets of measurements. In particular, we define the notion of locally-measured Rényi divergences, where the set of allowed measurements originates from locality constraints between (distant) parties A and B. As our main result, we derive variational characterizations of these locally-measured Rényi divergences. We then evaluate them for variants of data-hiding states, showcasing the reduced distinguishing power of locality-constrained measurements, and give corresponding applications in locally-measured hypothesis testing.**

I. Introduction

In the classical setting, Rényi divergences [1] are fundamental quantities in one-shot information theory with direct operational meaning in hypothesis testing. Hence, it is natural to investigate quantum extensions of these quantities. However, owing to the non-commutative nature of quantum physics, multiple possible extensions exist. Here, we follow a measurement-based approach to extend the classical Rényi divergence into the quantum domain. The particular method we use was previously applied to the relative entropy [2] and the Schatten one-norm [3]. The general idea works as follows: the measurement turns the given quantum states into classical measures to which we can apply the classical Rényi divergence. We then maximize the outcome over a set of allowed measurements. We focus our study on sets defined by various levels of locality constraints.

This is motivated by a problem in quantum hypothesis testing. Consider two distant laboratories which want to differentiate between two hypotheses modeled by bipartite quantum states. To achieve this, the laboratories conduct measurements on the states. However, the implementation of global measurements might not be feasible. Consequently, they are limited to local measurements, potentially supported by additional classical communication. Naturally, the question arises how well this restricted set performs compared to global measurements. A particular interesting set of states in this context are data-hiding states [4], [5]. These can be perfectly distinguished if all measurements are available but are almost indistinguishable using only local measurements and classical communication. This is an active area of research in the field of quantum information (cf. e.g. [3], [6]–[8] and references therein).

II. Definitions and General Properties

We first recall the definition of the classical Rényi divergence. Given two probability distributions μ and ν over a finite set $\mathcal{Z}$ with $\mu \ll \nu$, i.e. μ is absolutely continuous with respect to ν, we define Q_α for $\alpha \in (1,\infty)$ as

$$Q_\alpha\left(\mu\|\nu\right) := \sum_{z\in\mathcal{Z}} \mu(z)^\alpha \nu(z)^{1-\alpha} \tag{1}$$

and as $+\infty$ if $\mu \not\ll \nu$. With this, the Rényi divergence of order α [1] is defined as

$$D_\alpha\left(\mu\|\nu\right) := \frac{1}{\alpha-1}\log Q_\alpha(\mu\|\nu)\,. \tag{2}$$

For $\alpha \in (0,1)$, we use the same definitions, but Eq. (2) diverges only when μ is orthogonal to ν, denoted $\mu \perp \nu$. In the limit $\alpha \to 1$, we recover the Kullback-Leibler divergence or relative entropy [9] given for $\mu \ll \nu$ by

$$\lim_{\alpha\to 1} D_\alpha\left(\mu\|\nu\right) = D\left(\mu\|\nu\right) := \sum_{z\in\mathcal{Z}} \mu(z)\log\left(\frac{\mu(z)}{\nu(z)}\right) \tag{3}$$

and as $+\infty$ if $\mu \not\ll \nu$. For $\alpha \to \infty$, we similarly obtain the max-divergence defined as

$$\lim_{\alpha\to\infty} D_\alpha\left(\mu\|\nu\right) = D_{\max}\left(\mu\|\nu\right) := \sup_{z\in\mathcal{Z}} \log\left(\frac{\mu(z)}{\nu(z)}\right)\,. \tag{4}$$

In the following, we consider quantum systems of finite dimension $d \in \mathbb{N}$ to which we associate a Hilbert space $\mathcal{H}$. Let $\mathcal{P}$ denote the set of positive semi-definite (psd) operators acting on that space and $\mathcal{S}$ the subset of states, i.e. psd operators with unit trace. A general measurement on the quantum system is described by a positive operator-valued measure (POVM) M over a finite set $\mathcal{Z}$. Formally, M is a map from $\mathcal{Z}$ to $\mathcal{P}$ which satisfies $\sum_{z\in\mathcal{Z}} M^z = \mathbb{1}$. For a given $\rho \in \mathcal{P}$, the POVM induces a positive measure μ_ρ^M over $\mathcal{Z}$ according to the Born rule: $\mu_\rho^M(z) = \mathrm{tr}[\rho M^z]$.

For $\rho,\sigma \in \mathcal{S}$ with full support on $\mathcal{H}$, we then define the measured Rényi divergence of order $\alpha \in (0,1)\cup(1,\infty)$ as

$$D_\alpha^{\mathcal{M}}\left(\rho\|\sigma\right) := \sup_{M\in\mathcal{M}} D_\alpha\left(\mu_\rho^M \middle\| \mu_\sigma^M\right)\,, \tag{5}$$

with respect to a set of POVMs $\mathcal{M} \subseteq$ ALL. In an analogous manner, we define the measured relative entropy $D^{\mathcal{M}}$ (cf. [2])

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and measured max-divergence $D_{\max}^{\mathcal{M}}$. Similar to the classical quantity, we can rewrite the measured Rényi divergence as

$$D_\alpha^{\mathcal{M}}(\rho\|\sigma) = \frac{1}{\alpha-1}\log Q_\alpha^{\mathcal{M}}(\rho\|\sigma)\ , \tag{6}$$

where we defined $Q_\alpha^{\mathcal{M}}$ for $\alpha \in (0,1)$ as

$$Q_\alpha^{\mathcal{M}}(\rho\|\sigma) := \inf_{M\in\mathcal{M}} Q_\alpha\left(\mu_\rho^M \middle\| \mu_\sigma^M\right) \tag{7}$$

and for $\alpha \in (1,\infty)$ as the same expression but with the infimum replaced by a supremum. If ρ and σ do not have full support, we utilize the following expression to assign a value

$$\sup_{\epsilon\in(0,1]} D_\alpha^{\mathcal{M}}\left((1-\epsilon)\rho+\epsilon\pi\middle\|(1-\epsilon)\sigma+\epsilon\pi\right)\ , \tag{8}$$

where $\pi = \mathbb{1}/d$ is the completely mixed state. As before, analogous expressions are used for the relative entropy and max-divergence. These definitions are consistent at the extended orders $\alpha = 1$ and $\alpha = \infty$.

Lemma 1. *Let $\rho,\sigma \in \mathcal{S}$ and $\mathcal{M} \subseteq$ ALL. Then, the following limits hold*

$$D_1^{\mathcal{M}}(\rho\|\sigma) := \lim_{\alpha\to 1^-} D_\alpha^{\mathcal{M}}(\rho\|\sigma) = D^{\mathcal{M}}(\rho\|\sigma) \tag{9}$$

and

$$D_\infty^{\mathcal{M}}(\rho\|\sigma) := \lim_{\alpha\to\infty} D_\alpha^{\mathcal{M}}(\rho\|\sigma) = D_{\max}^{\mathcal{M}}(\rho\|\sigma)\ . \tag{10}$$

Proof. See full version [10]. □

Note that without imposing any restrictions on the set of POVMs, our definition reduces to the notion of measured Rényi divergence investigated in [11]. At $\alpha = 1$, we recover with $\mathcal{M} =$ ALL the notion of measured relative entropy studied by Donald [12] as well as Hiai and Petz [13].

Additionally, it is worth highlighting that D_α^{ALL} is in general not equal to a fully quantum Rényi divergence [11]. This differs from the measured Schatten one-norms from [3], where $\|\cdot\|_1 = \|\cdot\|_{\text{ALL}}$ holds. Nevertheless, there are two noteworthy exceptions to this rule. First, the measured Rényi divergence at order $\alpha = 1/2$ is connected to the measured fidelity via the relationship

$$D_{1/2}^{\mathcal{M}}(\rho\|\sigma) = -2\log F^{\mathcal{M}}(\rho,\sigma)\,, \tag{11}$$

where the measured fidelity is defined as

$$F^{\mathcal{M}}(\rho,\sigma) := \inf_{M\in\mathcal{M}} F\left(\mu_\rho^M,\mu_\sigma^M\right) \tag{12}$$

with $F(\mu,\nu) := Q_{1/2}(\mu\|\nu)$. It is known that $F = F_{\text{ALL}}$ holds for the fully-quantum version $F(\rho,\sigma) := \operatorname{tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right]$, which implies $D_{1/2} = D_{1/2}^{\text{ALL}}$. Second, the quantum max-divergence $D_{\max}(\rho\|\sigma) := \log\inf\{\lambda : \lambda\sigma \geq \rho\}$ [14] is known to be achievable by a measurement as well [15], i.e. we also have the equality $D_{\max} = D_{\max}^{\text{ALL}}$.

Before proceeding further, let us highlight some general properties shared by the measured Rényi divergence and their classical counterparts.

Lemma 2. *Let $\rho,\sigma \in \mathcal{S}$, $\mathcal{M} \subseteq$ ALL and $\alpha > 0$. Then, the following hold:*

1) $D_\alpha^{\mathcal{M}}(\rho\|\sigma) \geq 0$,
2) $D_\alpha^{\mathcal{M}}(\rho\|\sigma)$ *is a monotone increasing function in* α,
3) $Q_\alpha^{\mathcal{M}}(\rho\|\sigma)$ *is jointly concave in* (ρ,σ) *for* $\alpha < 1$ *and jointly convex for* $\alpha > 1$,
4) $D_\alpha^{\mathcal{M}}(\rho\|\sigma) < \infty$ *for* $\alpha \geq 1$ *if* $\rho \ll \sigma$ *and for* $\alpha \in (0,1)$ *if* $\rho \not\perp \sigma$.

Proof. See full version [10]. □

Note, however, that the measured Rényi divergence is in general not additive under tensor powers and not invariant under global unitary operations. Furthermore, it is worth highlighting that $\rho \perp \sigma$ does not necessarily imply that the measured Rényi divergences diverge to infinity. This showcases the phenomenon of quantum data hiding [4], [5]. We will demonstrate this in Sec. IV via an example.

In the following, we will focus exclusively on sets of measurements that are informationally complete [16]. These sets contain at least one POVM M which satisfies the condition that for $\rho \neq \sigma$ we have $\mu_\rho^M \neq \mu_\sigma^M$. This ensures that $D_\alpha^{\mathcal{M}}(\rho\|\sigma) > 0$ if $\rho \neq \sigma$ for $\rho,\sigma \in \mathcal{S}$, which allows us to interpret the measured Rényi divergence as a measure of distinguishability for quantum states.

Of particular interest are sets of POVMs that naturally arise in the context of bipartite states on systems A and B. The sets $\text{PPT}(A:B)$ and $\text{SEP}(A:B)$ contain POVMs whose elements have positive partial transpose or are separable, respectively. The set $\text{LOCC}(A:B)$ is operationally defined as all POVMs that can be implemented by local operations and finite classical communication. A special instance of this is $\text{LOCC}_1(A\to B)$, allowing only a single round of communication from A to B. Here, A measures her system, communicates the outcome to B, and, conditional on this outcome, B performs his measurement. Without loss of generality, we can write a POVM element of this class in the following form

$$M_{AB}^z = \sum_x M_A^x \otimes M_B^{z|x}\,. \tag{13}$$

Finally, if no communication is permitted between the two parties during measurement, we obtain the set $\text{LO}(A:B)$, consisting of product POVMs, i.e. elements of the form $M_{AB}^{x,y} = M_A^x \otimes M_B^y$. The following inclusions are trivial

$$\text{LO} \subseteq \text{LOCC}_1 \subseteq \text{LOCC} \subseteq \text{SEP} \subseteq \text{PPT} \subseteq \text{ALL}\,, \tag{14}$$

and all of them are known to be strict if the dimension of the system is large enough [3]. We refer to the induced Rényi divergences as locally-measured. Note, however, that the class $\text{PPT}(A:B)$ in general includes non-local, i.e. entangled POVMs. By the above inclusions, we immediately get the chain of inequalities

$$D_\alpha^{\text{LO}} \leq D_\alpha^{\text{LOCC}_1} \leq D_\alpha^{\text{LOCC}} \leq D_\alpha^{\text{SEP}} \leq D_\alpha^{\text{PPT}} \leq D_\alpha^{\text{ALL}}\,. \tag{15}$$

In Sec. IV, we show with an example that the gap between the locally-measured Rényi divergences and D_α^{ALL} can become arbitrarily large.

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III. Variational Formulas

Dual representations of divergences in terms of variational formulas serve as potent tools in their analysis. Motivated by this, we extend the results of [17] by deriving a generic upper bound on $D_\alpha^{\mathcal{M}}$ for arbitrary sets of measurements in terms of a variational formula. We further discuss when an exact dual representation can be obtained. To this end, let

$$C_{\mathcal{M}} := \bigcup_{M\in\mathcal{M}} \operatorname{cone}\{M^z\}_z \tag{16}$$

denote the union of the cones spanned by the POVM elements of the measurements in $\mathcal{M}$. With this definition in place, we state our main result.

Theorem 3. *For $\rho,\sigma\in\mathcal{S}$ and $\mathcal{M}\subseteq ALL$, we have*
1) for $\alpha\in(0,1/2)$:

$$Q_\alpha^{\mathcal{M}}(\rho\|\sigma) \geq \inf_{\omega\in C_{\mathcal{M}}} \alpha\operatorname{tr}[\rho\omega] + (1-\alpha)\operatorname{tr}\big[\sigma\omega^{\frac{\alpha}{\alpha-1}}\big] \tag{17}$$

$$= \inf_{\omega\in C_{\mathcal{M}}} \operatorname{tr}[\rho\omega]^\alpha \operatorname{tr}\big[\sigma\omega^{\frac{\alpha}{\alpha-1}}\big]^{1-\alpha}, \tag{18}$$

2) for $\alpha\in[1/2,1)$:

$$Q_\alpha^{\mathcal{M}}(\rho\|\sigma) \geq \inf_{\omega\in C_{\mathcal{M}}} \alpha\operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big] + (1-\alpha)\operatorname{tr}[\sigma\omega] \tag{19}$$

$$= \inf_{\omega\in C_{\mathcal{M}}} \operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big]^\alpha \operatorname{tr}[\sigma\omega]^{1-\alpha}, \tag{20}$$

and 3) for $\alpha\in(1,\infty)$:

$$Q_\alpha^{\mathcal{M}}(\rho\|\sigma) \leq \sup_{\omega\in C_{\mathcal{M}}} \alpha\operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big] + (1-\alpha)\operatorname{tr}[\sigma\omega] \tag{21}$$

$$= \sup_{\omega\in C_{\mathcal{M}}} \operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big]^\alpha \operatorname{tr}[\sigma\omega]^{1-\alpha}. \tag{22}$$

Proof. Here, we outline the sketch of the proof for the case $\alpha\in(1,\infty)$. For this, we follow the proof idea from [11] and use further modifications from [17, Lemma 3] to arrive at the claim. The other cases of α can be proven in an analogous fashion (see full version [10] for details).

Suppose first that both ρ and σ have full support. In this case, we have $\mu_\rho^M(z)/\mu_\sigma^M(z)\in(0,\infty)$. We define the shorthand $Q_\alpha^M(\rho\|\sigma) := Q_\alpha\big(\mu_\rho^M\big\|\mu_\sigma^M\big)$ for a fixed $M\in\mathcal{M}$ and then bound this as

$$Q_\alpha^M(\rho\|\sigma) = \sum_z \mu_\rho^M(z)\left(\frac{\mu_\rho^M(z)}{\mu_\sigma^M(z)}\right)^{\alpha-1} \tag{23}$$

$$= \operatorname{tr}\left[\rho\sum_z \sqrt{M^z}\left(\frac{\mu_\rho^M(z)}{\mu_\sigma^M(z)}\right)^{\alpha-1}\sqrt{M^z}\right] \tag{24}$$

$$\leq \operatorname{tr}\left[\rho\left(\sum_z\left(\frac{\mu_\rho^M(z)}{\mu_\sigma^M(z)}\right)^\alpha M^z\right)^{\frac{\alpha-1}{\alpha}}\right], \tag{25}$$

where the last line follows from an application of the operator Jensen inequality [18] together with the operator concavity of $f(t) := t^{\frac{\alpha-1}{\alpha}}$ [19]. For $M\in\mathcal{M}$, we further have by definition of $C_{\mathcal{M}}$ that

$$\omega_M := \sum_z\left(\frac{\mu_\rho^M(z)}{\mu_\sigma^M(z)}\right)^\alpha M^z \in C_{\mathcal{M}} \tag{26}$$

and with this we identify

$$\operatorname{tr}[\sigma\omega_M] = Q_\alpha^M(\rho\|\sigma)\,. \tag{27}$$

Thus, we obtain the desired bound:

$$Q_\alpha^{\mathcal{M}}(\rho\|\sigma) = \sup_{M\in\mathcal{M}} \alpha Q_\alpha^M(\rho\|\sigma) + (1-\alpha)Q_\alpha^M(\rho\|\sigma) \tag{28}$$

$$\leq \sup_{\omega\in C_{\mathcal{M}}} \alpha\operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big] + (1-\alpha)\operatorname{tr}[\sigma\omega]\,. \tag{29}$$

This can be extended to the general case of states without full support via Eq. (8). Lastly, for the equivalence of the variational expressions, we use an idea in [11] and employ Bernoulli's inequality to obtain

$$\sup_{\omega\in C_{\mathcal{M}}} \alpha\operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big] + (1-\alpha)\operatorname{tr}[\sigma\omega] \tag{30}$$

$$\leq \sup_{\omega\in C_{\mathcal{M}}} \operatorname{tr}[\sigma\omega]\left(1+\operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big]\operatorname{tr}[\sigma\omega]^{-1}-1\right)^\alpha \tag{31}$$

$$= \sup_{\omega\in C_{\mathcal{M}}} \operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big]^\alpha \operatorname{tr}[\sigma\omega]^{1-\alpha}\,. \tag{32}$$

For the reverse direction, we first note that for any $\omega\in C_{\mathcal{M}}$ and any $\lambda>0$, we also have $\lambda\omega\in C_{\mathcal{M}}$ by definition of $C_{\mathcal{M}}$. Thus, we can rewrite the variational expression as

$$\sup_{\substack{\omega\in C_{\mathcal{M}}\\ \operatorname{tr}[\omega]=1}} \sup_{\lambda>0} \alpha\operatorname{tr}\Big[\rho(\lambda\omega)^{\frac{\alpha-1}{\alpha}}\Big] + (1-\alpha)\operatorname{tr}[\sigma\lambda\omega]\,. \tag{33}$$

Picking $\lambda^\star := \operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big]^\alpha \operatorname{tr}[\sigma\omega]^{-\alpha} > 0$, we obtain a lower-bound on the variational expression in terms of

$$\sup_{\substack{\omega\in C_{\mathcal{M}}\\ \operatorname{tr}[\omega]=1}} \operatorname{tr}\Big[\rho\omega^{\frac{\alpha-1}{\alpha}}\Big]^\alpha \operatorname{tr}[\sigma\omega]^{1-\alpha}\,. \tag{34}$$

Since the multiplicative expression is scaling-invariant, the trace condition can be dropped and we get the desired claim. For the orders $\alpha\in(0,1)$, the arithmetic-geometric mean inequality is used instead of Bernoulli's inequality. □

By the definition of $Q_\alpha^{\mathcal{M}}$, we thus obtain an upper bound on $D_\alpha^{\mathcal{M}}$ for all values of α. Note that for $\alpha\in(0,1)$ the minimization is over a convex function in ω. Correspondingly, for $\alpha\in(1,\infty)$ the maximization is over a concave function.

Observe that, C_{ALL} is the cone of positive semi-definite operators and for this case the given upper bounds are known to become an equality [11]. However, for other sets of measurements this is in general not clear and a good characterization of the set $C_{\mathcal{M}}$ is not always available. Nevertheless, we can write down an exact representation of the measured Rényi divergence in terms of a variational expression for the cases of $\mathrm{LO}(A:B)$ and $\mathrm{LOCC}_1(A\to B)$. Similar as in the proof of [17, Lemma 4] for the locally-measured relative entropy, this can be achieved by going to a larger Hilbert space. Thus, let us introduce the spaces A' isomorphic to $A\otimes A$ and B' isomorphic to $B\otimes B$. We can then define the set of operators

$$C^\star_{\mathrm{LO}(A':B')} := \left\{\sum_{x=1}^{d_A^2}\sum_{y=1}^{d_B^2} c_{x,y}P_{A'}^x\otimes P_{B'}^y\right\}, \tag{35}$$

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where $P^x_{A'}, P^y_{B'}$ are sets of orthogonal projectors and $c_{x,y} \geq 0$. These are operators that are classical in some basis on A' and B'.

Proposition 4. *Let $\rho_{A'B'}$ and $\sigma_{A'B'}$ be consistent embeddings of ρ_{AB} and σ_{AB} in $A' \otimes B'$ as defined above, respectively. Then, the inequalities for $Q^{LO}_\alpha(\rho_{AB}\|\sigma_{AB})$ in terms of the variational expressions given in Theorem 3 become an equality when the following replacements are made: $\rho \mapsto \rho_{A'B'}$, $\sigma \mapsto \sigma_{A'B'}$ and $C_{LO(A:B)} \mapsto C^\star_{LO(A':B')}$.*

Proof. See full version [10]. □

For the class $\mathrm{LOCC}_1(A \to B)$ the proposition holds as written above if we use consistent embeddings $\rho_{A'B}$ and $\sigma_{A'B}$ in $A' \otimes B$. Further, we have to replace $C_{\mathrm{LOCC}_1(A\to B)}$ by the set

$$C^\star_{\mathrm{LOCC}_1(A'\to B)} := \left\{ \sum_{x=1}^{d_A^2} P^x_{A'} \otimes \omega^x_B \right\}, \tag{36}$$

where $P^x_{A'}$ denotes a complete set of orthogonal projectors and $\omega^x_B \in \mathcal{P}$. These are operators that are classical-quantum in some basis on A'.

For the sets $\mathrm{LOCC}(A:B)$, $\mathrm{SEP}(A:B)$ and $\mathrm{PPT}(A:B)$ we do not know if equality always holds in Theorem 3. However, the sets C_{SEP} and C_{PPT} become simple and just consist of separable positive semi-definite operators and positive semi-definite operators with positive partial transpose, respectively.

Let us take a closer look at the variational characterization for special orders α. First, in the limit $\alpha \to 1^-$, our result reproduces the generic bound obtained in [17, Lemma 3] for the measured relative entropy

$$D^{\mathcal{M}}(\rho\|\sigma) \leq \sup_{\omega\in C_{\mathcal{M}}} \mathrm{tr}[\rho\log\omega] - \log\mathrm{tr}[\sigma\omega] \tag{37}$$

$$= \sup_{\omega\in C_{\mathcal{M}}} \mathrm{tr}[\rho\log\omega] + 1 - \mathrm{tr}[\sigma\omega]\,. \tag{38}$$

Further, using the connection of the measured Rényi divergence at $\alpha = 1/2$ to the measured fidelity (cf. Sec. II), we obtain a generalization of Alberti's theorem [20] as

$$F_{\mathcal{M}}(\rho,\sigma) \geq \inf_{\omega\in C_{\mathcal{M}}} \sqrt{\mathrm{tr}[\rho\omega^{-1}]\,\mathrm{tr}[\sigma\omega]}\,. \tag{39}$$

Finally, in the limit $\alpha \to \infty$, we get

$$D^{\mathcal{M}}_{\max}(\rho\|\sigma) \leq \sup_{\omega\in C_{\mathcal{M}}} \log\left(\frac{\mathrm{tr}[\rho\omega]}{\mathrm{tr}[\sigma\omega]}\right). \tag{40}$$

Here, we know that equality holds for the sets $\mathrm{SEP}(A:B)$ and $\mathrm{PPT}(A:B)$.

IV. Application: Data Hiding

As an application of our variational characterizations, we use them to compute the locally-measured Rényi divergences for examples of data hiding states. As mentioned in Sec. II, these states can be perfectly distinguished by a global measurement, yet they are almost indistinguishable when using only local measurements. Here, we aim to quantify this data hiding effect by means of the locally-measured Rényi divergences and connect this to the task of locally-measured hypothesis testing.

We look at the maximally entangled state Φ and test it against its orthogonal complement $\Phi^\perp$, which are given by

$$\Phi = \frac{1}{d}\sum_{i,j} |ii\rangle\langle jj|\,, \qquad \Phi^\perp = \frac{\mathbb{1}-\Phi}{d^2-1}\,, \tag{41}$$

where $\{|ij\rangle\}_{i,j}$ denotes an orthonormal basis (ONB) for $A \otimes B$ and both systems have dimension d. This example was previously studied in [6], [21], [22]. Notice that if we are allowed to use all measurements we can perform the test $T := \{\Phi, \mathbb{1}-\Phi\}$ to obtain

$$\mu^T_\Phi = \{1,0\}\,, \qquad \mu^T_{\Phi^\perp} = \{0,1\}\,, \tag{42}$$

and therefore $D^{\mathrm{ALL}}_\alpha\left(\Phi\middle\|\Phi^\perp\right) = \infty$ for all orders $\alpha > 0$. However, it is important to notice that the test T employs the POVM element Φ which has a negative partial transpose and as such is not in any of the classes introduced in Sec. II. It turns out that if we restrict the available measurements locally to different smaller sets all of them perform equally poorly.

Proposition 5. *For $\mathcal{M} = \{LO, LOCC_1, LOCC, SEP, PPT\}$ and all orders $\alpha > 0$, we have*

$$D^{\mathcal{M}}_\alpha\left(\Phi\middle\|\Phi^\perp\right) = \log(d+1)\,. \tag{43}$$

Proof. Note that the case $\alpha = 1$ was previously proven in [22]. The general idea of the proof is to lower-bound D^{LO}_α and then to find a matching upper-bound via D^{PPT}_α. This is the same meta idea that references [6], [21], [22] followed.

Let us start with the lower bound on D^{LO}_α. For this, we measure the states with POVM elements of the form $M^{(i,j)} := |i\rangle\langle i| \otimes |j\rangle\langle j|$. Clearly, we have $M \in \mathrm{LO}$. By straightforward computation, we get our desired lower bound for all $\alpha > 0$,

$$D^{\mathrm{LO}}_\alpha\left(\Phi\middle\|\Phi^\perp\right) \geq D_\alpha\left(\mu^M_\Phi\middle\|\mu^M_{\Phi^\perp}\right) = \log(d+1)\,. \tag{44}$$

For the second part, we employ our variational characterization to bound D^{PPT}_α. We only need to consider the case $\alpha = \infty$ due to the monotonicity in α. Here, we showed that the following holds

$$D^{\mathrm{PPT}}_{\max}(\Phi\|\Phi^\perp) = \sup_{\omega\in C_{\mathrm{PPT}}} \log\left(\frac{\mathrm{tr}[\Phi\omega]}{\mathrm{tr}[\Phi^\perp\omega]}\right). \tag{45}$$

Recall that the set C_{PPT} is comprised of positive semi-definite operators with a positive partial transpose. Further, observe that the problem is scaling invariant so we can limit the optimization to $\omega \in \mathcal{S}$ with the PPT property.

Next, the two given states under consideration are special cases of so-called isotropic states, which take the general form

$$i(p) = p\Phi + (1-p)\Phi^\perp \tag{46}$$

and have the PPT property for $p \in \left(0, \frac{1}{d}\right]$ [23]. These states are characterized by their invariance under conjugation with $U \otimes \bar{U}$, where U is a unitary operator and the bar denotes complex conjugation. We can exploit this symmetry similarly as in [6], [21], [22]. Namely, we have that $\mathrm{tr}[i(p)\omega] = \mathrm{tr}[i(p)\mathcal{I}(\omega)]$ for

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any $\omega \in \mathcal{P}$, whereby $\mathcal{I}$ denotes the isotropic twirling whose action on any operator X is given by

$$\mathcal{I}(X) = \int_{\mathcal{U}(d)} (U \otimes \bar{U}) X (U \otimes \bar{U})^\dagger \, \mathrm{d}\mu_H(U) \,, \tag{47}$$

where the integration is with respect to the Haar measure on the unitary group $\mathcal{U}(d)$.

After twirling, the objective function is given solely in terms of $\mathcal{I}(\omega)$. Since $\mathcal{I}$ preserves the PPT property of ω, $\mathcal{I}(\omega)$ is an isotropic PPT state characterized completely by a single parameter c (cf. Eq. (46)). This allows us to rewrite the optimization problem into

$$\text{maximize} \;\; \log\left(\frac{c}{1-c}\right) + \log\left(d^2-1\right) \;\; \text{s.t. } 0 \leq c \leq \frac{1}{d} \,. \tag{48}$$

This has the solution $\log(d+1)$ with which we obtain the desired upper bound. □

This proof technique can in principle also be used to compute the case where $n > 1$ copies of the given states are tested against each other. However, this becomes rather tedious as the number of copies grows and therefore we show in the following an alternative route to compute the n-copy case. We prepare the proof with the following lemma that gives a dual characterization for $D_{\max}^{\text{PPT}}$. Note that this has a similar form to the dual program of the PPT-measured norm derived in [6].

Lemma 6. *For $\rho, \sigma \in \mathcal{S}$, we have*

$$D_{\max}^{PPT}(\rho\|\sigma) = \log \inf_{\substack{\lambda>0 \\ A,B\in\mathcal{P}}} \left\{\lambda : A + B^\Gamma = \lambda\sigma - \rho\right\}. \tag{49}$$

Proof. We start by observing that we can rewrite the variational program into

$$\sup_{\omega\in C_{\text{PPT}}} \log\left(\frac{\operatorname{tr}[\rho\omega]}{\operatorname{tr}[\sigma\omega]}\right) = \sup_{\omega\in C_{\text{PPT}}} \left\{\log \operatorname{tr}[\rho\omega] : \operatorname{tr}[\sigma\omega] \leq 1\right\}. \tag{50}$$

Using this reformulation, we proceed to find its dual program with the Lagrangian formalism. Observe that the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \operatorname{tr}[\rho\omega] + \operatorname{tr}[A\omega] + \operatorname{tr}\left[B\omega^\Gamma\right] + \lambda\left(1 - \operatorname{tr}[\sigma\omega]\right) \tag{51}$$

$$= \lambda + \operatorname{tr}\left[(\rho - \lambda\sigma + A + B^\Gamma)\omega\right], \tag{52}$$

with the dual variables $A, B \in \mathcal{P}$ and $\lambda \in \mathbb{R}_+$. For the dual program, this has to be maximized over all Hermitian operators ω without any constraints. The result is infinity unless $\rho - \lambda\sigma + A + B^\Gamma = 0$. In such a case, the maximum is simply given by λ. Thus, we can conclude that the dual program is given by

$$\inf_{\substack{\lambda>0 \\ A,B\in\mathcal{P}}} \left\{\log\lambda : A + B^\Gamma = \lambda\sigma - \rho\right\}. \tag{53}$$

Since Slater's condition is satisfied (e.g. $\omega^\star = \pi/2$ is strictly feasible), strong duality holds. □

The dual variational characterization of $D_{\max}^{\text{PPT}}(\rho\|\sigma)$ enables us to compute the n-copy case in a more elegant way. The following proposition states our result.

Proposition 7. *For $\mathcal{M} = \{LO, LOCC_1, LOCC, SEP, PPT\}$ and all orders $\alpha > 0$, we have*

$$D_\alpha^{\mathcal{M}}\left(\Phi^{\otimes n}\middle\|\left(\Phi^\perp\right)^{\otimes n}\right) = n\log(d+1). \tag{54}$$

That is, the locally-measured Rényi divergences are additive under tensor powers for this example.

Proof. For the lower bound, observe that measuring with the product POVM $M^{\otimes n}$ where $M^{(i,j)} = |i\rangle\langle i| \otimes |j\rangle\langle j|$ gives the lower bound $n\log(d+1)$ for D_α^{LO}.

For the upper bound, we use the dual characterization of $D_{\max}^{\text{PPT}}$ and that we can pick any dual-feasible point to obtain an upper bound. The point we pick is $\lambda = (d+1)^n$, $A = 0$ and

$$B = \frac{1}{(d-1)^n}\left[\left(\mathbb{1} - \frac{F}{d}\right)^{\otimes n} - (d-1)^n\left(\frac{F}{d}\right)^{\otimes n}\right], \tag{55}$$

where $F := \sum_{i,j} |ij\rangle\langle ji|$ denotes the swap operator. To see that this is indeed dual feasible, observe that B has eigenvalues

$$\frac{1}{(d-1)^n}\left[\left(1 \mp \frac{1}{d}\right)^n - (d-1)^n\left(\frac{\pm 1}{d}\right)^n\right] \geq 0. \tag{56}$$

and that $B^\Gamma = \lambda\left(\Phi^\perp\right)^{\otimes n} - \Phi^{\otimes n}$. With this, we obtain the matching upper bound $n\log(d+1)$. □

This gives an alternative way to prove a result in [6], where the optimal Stein's exponent for locally-measured hypothesis testing $\zeta_{\text{Stein}}^{\mathcal{M}}(\Phi, \Phi^\perp, \varepsilon) = \log(d+1)$ was computed for $\mathcal{M} = \{\text{LO}, \text{LOCC}_1, \text{LOCC}, \text{SEP}, \text{PPT}\}$. Whereas our α-bounds from Proposition 7 then allow to derive the hypothesis testing results from [6] in a simple manner (see the full version [10]). There is the even more direct connection to the Stein's lemma for restricted measurements from [7]. It was shown that

$$\lim_{\varepsilon\to 0} \zeta_{\text{Stein}}^{\mathcal{M}}(\rho, \sigma, \varepsilon) = \lim_{n\to\infty} \frac{1}{n} D^{\mathcal{M}}\left(\rho^{\otimes n}\middle\|\sigma^{\otimes n}\right), \tag{57}$$

which immediately becomes single-letter for $\rho = \Phi$, $\sigma = \Phi^\perp$.

V. Conclusion

We have introduced the notion of locally-measured Rényi divergence as a quantum extension of classical Rényi divergence. As our main result, we derived variational characterizations of these quantities. This tool gives a novel approach to prove operational results in quantum information theory whenever the applied operations have some locality constraints, e.g., for quantum hypothesis testing under restricted measurements. In order to showcase the practical application, we computed the locally-measured Rényi divergences for an example of data hiding states. Looking ahead, we will fully work out the connection to locally-measured hypothesis testing.

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