Entropy for multipartite quantum states

Mario Berta

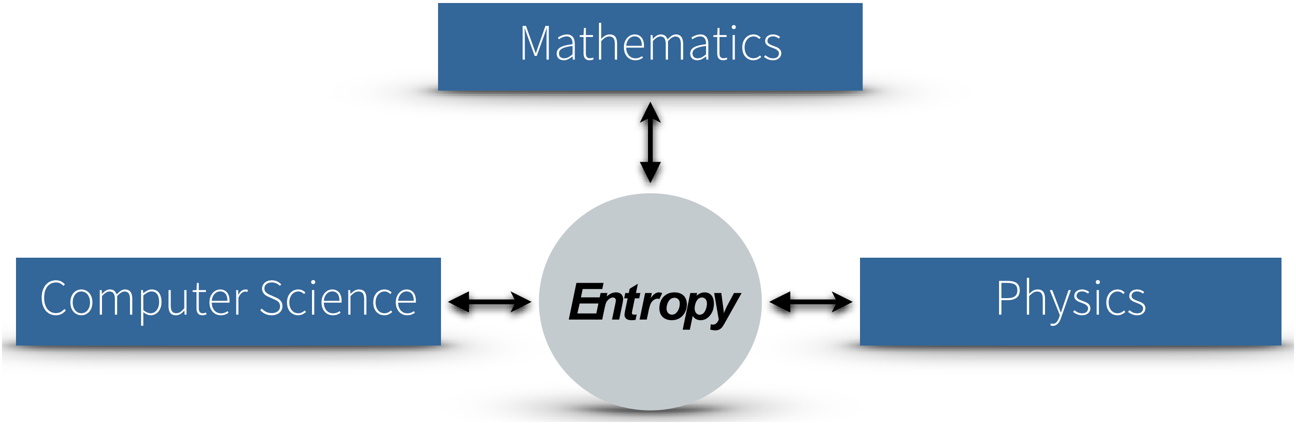
Institute for Quantum Information
Department of Physics

RWTH AACHEN UNIVERSITY

From classical to quantum entropy inequalities

Entropy function

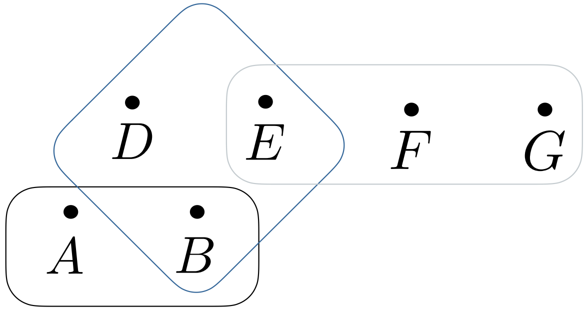
- Probability distribution P_A with Shannon entropy $H(A)_P := -\sum_x P_A^x \log P_A^x \geq 0$



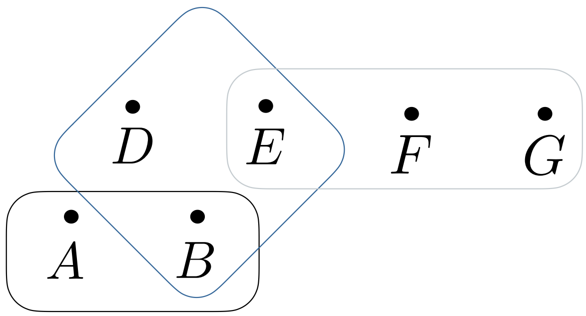
Interdisciplinary concept

- Focus on mathematical properties of entropy

  → correlation measures for P_{ABC} multipartite distributions?



Standard inequalities $[\alpha]$



- Given multipartite probability distribution P_{ABC}, define
  - Mutual information:

$$I(A:B)_P := H(A)_P + H(B)_P - H(AB)_P \geq 0\ [\alpha_{\mathrm{SA}}] \text{ sub-additivity (SA)}$$

  - Conditional entropy:

$$H(A|B)_P := H(AB)_P - H(B)_P \geq 0\ [\alpha_{\mathrm{C}}]$$

  - Conditional mutual information:

$$I(A:B|C)_P := H(AC)_P + H(BC)_P - H(C)_P - H(ABC)_P \geq 0\ [\alpha_{\mathrm{SSA}}]$$

strong sub-additivity (SSA)

- Multipartite extensions possible, similar

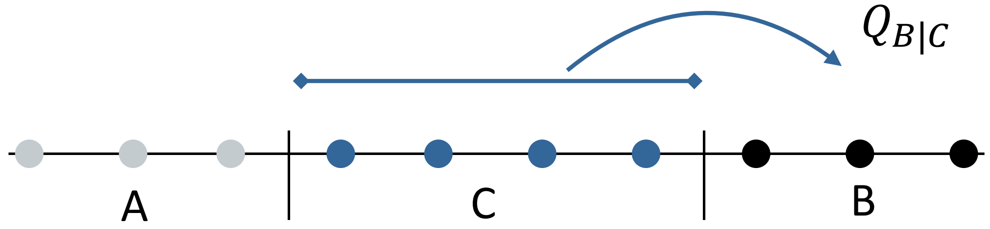
Refined inequalities [β]

- Parent quantity Kullback-Leibler (KL) divergence

$$D_{\mathrm{KL}}(P||Q) \coloneqq \sum_x P^x \log\left(\frac{P^x}{Q^x}\right) \geq 0 \text{ with equality condition } D_{\mathrm{KL}}(P||Q) = 0 \Leftrightarrow P = Q$$

- Distance to product distributions $I(A{:}B)_P = \inf_{Q_A \times Q_B} D_{\mathrm{KL}}(P_{AB}||Q_A \times Q_B) \geq 0$ [β_{P}]

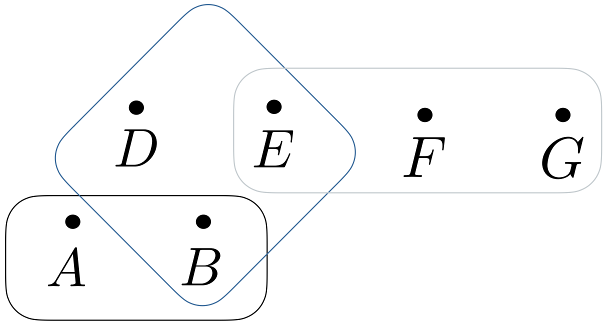
- Local recoverability $I(A{:}B|C)_P = \inf_{Q_{B|C}} D_{\mathrm{KL}}\left(P_{ABC}||Q_{B|C}P_{AC}\right) \geq 0$ [β_{LR}]



- Markov chain formulation (equivalent) $I(A{:}B|C)_P = \inf_{Q_{A-C-B}} D_{\mathrm{KL}}(P_{ABC}||Q_{A-C-B}) \geq 0$ [β_{MC}]

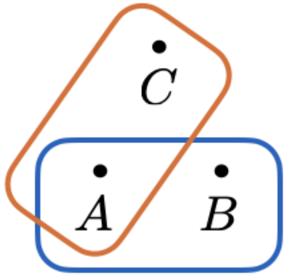
Are there quantum versions?

Quantum entropy & entanglement



- Classical versus quantum correlations (aka entanglement)?
- Example: Two maximally quantumly correlated qubits A and B *cannot* be quantumly correlated with any C

  → monogamy of entanglement



- Motivation: Understanding which classical entropy inequalities can be lifted to quantum entropy inequalities ⇔ entanglement structure of quantum states

Quantum entropy

Finite-dimensional quantum information

- Quantum systems are inner product spaces $\mathcal{H}$, with tensor products $\mathcal{H}_{AB} \coloneqq \mathcal{H}_A \otimes \mathcal{H}_B$
- Quantum states are elements in $\mathcal{S}(\mathcal{H}) \coloneqq \{\rho \in \mathrm{Lin}(\mathcal{H}) \mid \rho \geq 0 \vee \mathrm{Tr}[\rho] = 1\}$
- Quantum channels are elements in

  $Q_{\mathrm{ch}}(\mathcal{H} \to \mathcal{H}') \coloneqq \{\mathcal{N}: \mathrm{Lin}(\mathcal{H}) \to \mathrm{Lin}(\mathcal{H}') \mid \mathcal{N} \text{ completely positive \& trace preserving}\}$

- Example: Partial trace $\mathrm{Tr}[\mathrm{Tr}_B[X_{AB}]Y_A] = \mathrm{Tr}[X_{AB}(Y_A \otimes 1_B)]\ \forall\, Y_A \in \mathrm{Lin}(\mathcal{H}_A)$

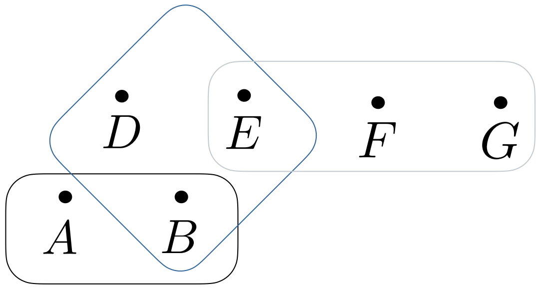
  → write $\mathrm{Tr}_B[\rho_{AB}] \eqqcolon \rho_A$ and abbreviate $(\rho_A \otimes 1_B)\rho_{AB}(1_A \otimes \rho_B) \eqqcolon \rho_A \rho_{AB} \rho_B$

- Example: Measurement $\mathbb{P}_{\rho,M}(x) = \mathrm{Tr}[\rho M^x]$ via $\{M^x \geq 0\}_x$ with $\sum_x M^x = 1$ POVM

Standard quantum inequalities $[\alpha]$

- Quantum state ρ_A with von Neumann entropy

$$H(A)_\rho := -\mathrm{Tr}[\rho_A \log \rho_A] \geq 0$$

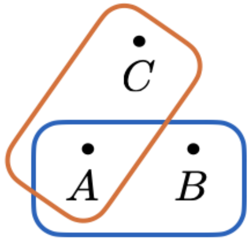


- $I(A:B)_\rho := H(A)_\rho + H(B)_\rho - H(AB)_\rho \geq 0\ [\alpha_{\mathrm{QSA}}]$
- $H(A|B)_\rho := H(AB)_\rho - H(B)_\rho$ negative for some entangled states → no $[\alpha_{\mathrm{QC}}]$
- Strong sub−additivity (SSA) due to entanglement monogamy [Lieb & Ruskai LMP 1973]

$$I(A:B|C)_\rho := H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho \geq 0\ [\alpha_{\mathrm{QSSA}}]$$

- Intuition: SSA equivalent to weak monotonicity

$$H(A|C)_\rho + H(A|B)_\rho \geq 0\ [\alpha_{\mathrm{QWM}}] \Leftrightarrow [\alpha_{\mathrm{QSSA}}]$$



What about refined inequalities?

Quantum relative entropy

- Classical KL divergence $D_{\text{KL}}(P||Q) = \sum_x P^x \log\left(\frac{P^x}{Q^x}\right)$ for refined inequalities $[\beta]$
- Quantum relative entropy [Umegaki TMJ 1954] [Hiai & Petz CMP 1991]

$$D(\rho||\sigma) = \text{Tr}[\rho(\log(\rho) - \log(\sigma))]$$

with (KL): simplifies to KL divergence via $\rho_C = \text{diag}(\{P^x\}), \sigma_C = \text{diag}(\{Q^x\})$

with (MONO): monotonicity under quantum channels $\mathcal{N}$ as

$$D(\rho||\sigma) \geq D\big(\mathcal{N}(\rho)||\mathcal{N}(\sigma)\big)$$

- Question: Unique quantum extension with (KL) + (MONO)? If not, which one(s) to use?

Classical to quantum relative entropy

- Classical $D_{\mathrm{KL}}(P||Q) = \sum_x P^x \log\left(\frac{P^x}{Q^x}\right)$ and quantum $D(\rho||\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$
- Geometric relative entropy [Belavkin-Staszewski AHP 1982]

$$\widehat{D}(\rho||\sigma) := \mathrm{Tr}\left[\rho\log\left(\rho^{1/2}\sigma^{-1}\rho^{1/2}\right)\right]$$

- Measured relative entropy via $\{M^x\}_x$ with $\mathbb{P}_{\rho,M}(x) = \mathrm{Tr}[M^x\rho]$ as [Donald CMP 1986]

$$D_{\mathrm{ALL}}(\rho||\sigma) := \sup_M D_{\mathrm{KL}}\left(\mathbb{P}_{\rho,M}||\mathbb{P}_{\sigma,M}\right)$$

- Theorem: $\widehat{D}(\rho||\sigma) \geq D(\rho||\sigma) \geq D_{\mathrm{ALL}}(\rho||\sigma)$ + equality iff commute [Hiai & Petz CMP 1991]
- Theorem: All quantum extensions with (KL) + (MONO) are in between [Matsumoto 2015]

Further: Locally measured relative entropies

- Bipartite system AB, local measurements with respect to $A{:}\,B$ as

$$\mathrm{M} = \mathrm{LO}(A{:}\,B), \mathrm{LOCC}_1(\mathrm{A} \to B), \mathrm{LOCC}(A{:}\,B), \mathrm{SEP}(A{:}\,B), \mathrm{PPT}(A{:}\,B), \mathrm{ALL}$$

leads with $\mathbb{P}_{\rho,M}(x) = \mathrm{Tr}[\rho M^x]$ from $\{M^x\}_x$ with $M^x \in \mathrm{M}$ to definition [Piani PRL 2009]

$$D_{\mathrm{M}}(\rho_{AB}||\sigma_{AB}) \coloneqq \sup_{M \in \mathrm{M}} D_{\mathrm{KL}}(\mathbb{P}_{\rho,M}||\mathbb{P}_{\sigma,M})$$

- Classical special case (KL), adapted monotonicity (MONO’), ordering

$$D_{\mathrm{ALL}}(\rho_{AB}||\sigma_{AB}) \geq D_{\mathrm{PPT}}(\rho_{AB}||\sigma_{AB}) \geq \cdots \geq D_{\mathrm{LO}}(\rho_{AB}||\sigma_{AB}) \geq 0$$

and informationally complete $D_{\mathrm{M}}(\rho_{AB}||\sigma_{AB}) = 0 \Leftrightarrow \rho_{AB} = \sigma_{AB}$

Mathematical tools for quantum entropy

Quantum entropy & Matrix analysis

- Multipartite systems ABC $\Leftrightarrow$ non-commuting matrices $\rho_{ABC}, \rho_{AB}, \rho_{BC}, \cdots$
- Matrix trace inequalities, e.g., for H_1, H_2 Hermitian:

  i. $\exp(H_1)\exp(H_2) = \exp(H_1 + H_2 + 0.5 \cdot [H_1, H_2] + \cdots)$ Baker-Campbell-Hausdorf

  ii. $\mathrm{Tr}[\exp(H_1)\exp(H_2)] \geq \mathrm{Tr}[\exp(H_1 + H_2)]$ Golden-Thompson

  iii. $\exp(\mathrm{Tr}[\exp(H_1)H_2]) \leq \mathrm{Tr}[\exp(H_1 + H_2)]$ for $\mathrm{Tr}[\exp(H_1)] = 1$ Peierls-Bogolyubov

- Example: Entropy is non-negative

$$H(A)_\rho = -\mathrm{Tr}[\rho_A \log(\rho_A)] \geq \log \mathrm{Tr}\big[\exp\big(\log(\rho_A) + \log(\rho_A)\big)\big] = -\log \mathrm{Tr}[\rho_A^2] \geq 0$$

- Challenge: Multipartite quantum systems $\Leftrightarrow$ Multivariate trace inequalities

Workhorse A: Multivariate trace inequalities

- Multipartite systems ABC $\Leftrightarrow$ non-commuting matrices $\rho_{ABC}, \rho_{AB}, \rho_{BC}, \cdots$
- Example trace inequality: Golden-Thompson for H_1, H_2 Hermitian as

$$\mathrm{Tr}[\exp(H_1 + H_2)] \leq \mathrm{Tr}[\exp(H_1)\exp(H_2)]$$

- Theorem: Multivariate extension for $\{H_k\}_{k=1}^{n}$ Hermitian and $p \geq 1$ as

$$\left\| \exp\left(\sum_{k=1}^{n} H_k\right) \right\|_{\mathrm{p}} \leq \int_{-\infty}^{\infty} \beta_0(t) \left\| \prod_{k=1}^{n} \exp\big((1+it)H_k\big) \right\|_{\mathrm{p}}$$

with probability distribution $\beta_0(t) := \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}$ [Sutter, **B.**, Tomamichel CMP 2017]

- Proof: Via multivariate Araki-Lieb-Thirring using interpolation theory in complex analysis

Workhorse A: Example

- Corollary: For $n = 2$ and $p = 2$ relaxed to Golden-Tompson inequality

$$\mathrm{Tr}[\exp(H_1 + H_2)] \leq \mathrm{Tr}[\exp(H_1)\exp(H_2)]$$

- Corollary: For $n = 3$ and $p = 2$ relaxed to Lieb's triple matrix inequality [Lieb AM 1973]

$$\mathrm{Tr}[\exp(H_1 + H_2 + H_3)] \leq \int_{-\infty}^{\infty} \mathrm{d}\beta_0(t)\mathrm{Tr}[\exp(H_1)\exp\left(\tfrac{1+it}{2} H_2\right) \exp(H_3)\exp\left(\tfrac{1-it}{2} H_2\right)]$$

$$= \int_0^{\infty} \mathrm{d}\lambda \, \mathrm{Tr}\left[\exp(H_1)\frac{1}{\exp(-H_2+\lambda)} \exp(H_3) \, \frac{1}{\exp(-H_2+\lambda)}\right]$$

cf. [Lemm JMP 2018]

- Arbitrarily many matrices extension allows to treat multipartite quantum states

Workhorse B: Variational formulae

- Kullback-Leibler divergence

$$D_{\mathrm{KL}}(P||Q) = \sup_{W \geq 0} \sum_x P_x \log W_x - \log \sum_x Q_x W_x$$

- Theorem: Quantum relative entropy [Araki] [Petz CMP 1988]

$$D(\rho||\sigma) = \sup_{\omega \geq 0} \mathrm{Tr}[\rho \log \omega] - \log \mathrm{Tr}[\exp(\log \sigma + \log \omega)]$$

- Theorem: Measured relative entropy [Donald CMP 1986] [**B.**, Fawzi, Tomamichel LMP 2017]

$$D_{\mathrm{ALL}}(\rho||\sigma) = \sup_{\omega \geq 0} \mathrm{Tr}[\rho \log \omega] - \log \mathrm{Tr}[\sigma \omega]$$

- Question: What about other non-commutative extensions?

Workhorse B: Continued

- Unclear for the geometric relative entropy $\widehat{D}(\rho||\sigma)$ [Ando & Hiai LAA 1994]
- Theorem: Locally measured relative entropies [**B.** & Tomamichel CMP 2024]

$$D_{\mathrm{M}}(\rho||\sigma) \leq \sup_{\omega \in C_M} \mathrm{Tr}[\rho \log \omega] - \log \mathrm{Tr}[\sigma\omega]$$

for the different sets M with above supremum restricted to $C_{\mathrm{M}} \coloneqq \bigcup_{M \in \mathrm{M}} \mathrm{cone}\{M_x\}$

- Corollary: [Rippchen, Sreekumar, **B.** IEEE 2025]
  - Equality for $\mathrm{M} = \mathrm{LO}, \mathrm{LOCC}_1\ (A \to B)$ via corresponding embeddings based on Naimark's theorem
  - Otherwise, there is in general a gap
- Proof: Apply operator Jensen's inequality

Tools in action: First proof

Quantum strong sub-additivity $[\alpha_{\mathrm{QSSA}}]$

- Claim: $I(A:B|C)_\rho = H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho \geq 0$
- Proof classical: $I(A:B|C)_{\mathrm{P}} = D_{\mathrm{KL}}(P_{ABC}||P_{AC}P_{BC}P_C^{-1}) = D_{\mathrm{KL}}\left(P_{ABC}||P_{AC}P_{B|C}\right) \geq 0$
- Proof quantum: [Sutter, **B.**, Tomamichel CMP 2017] [**B.**, Hirche, Brandão CMP 2021] [**B.**, Sutter, Walter CMP 2023]

$$I(A:B|C)_\rho = D(\rho_{ABC}||\exp(\log\rho_{AC} + \log\rho_{BC} - \log\rho_{ABC}))$$

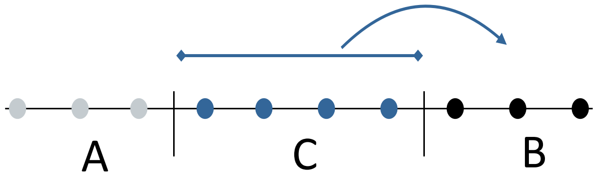
$$= \sup_{\omega_{ABC}\geq 0} \mathrm{Tr}[\rho_{ABC}\log\omega_{ABC}] - \log\mathrm{Tr}[\exp(\log\omega_{ABC} + \log\rho_{AC} + \log\rho_{BC} - \log\rho_C)]$$

$$\geq \sup_{\omega_{ABC}\geq 0} \mathrm{Tr}[\rho_{ABC}\log\omega_{ABC}] - \log\mathrm{Tr}[\omega_{ABC}\sigma_{ABC}] = D_{\mathrm{ALL}}(\rho_{ABC}||\sigma_{ABC}) \geq 0 \quad \text{qed}$$

for $\sigma_{ABC} = \int_{-\infty}^{\infty} \mathrm{d}\beta_0(t)\, \rho_{BC}^{\frac{1+it}{2}} \rho_C^{\frac{-1+it}{2}} \rho_{AC} \rho_C^{\frac{-1-it}{2}} \rho_{BC}^{\frac{1-it}{2}} \geq 0$ with $\mathrm{Tr}[\sigma_{ABC}] = 1$

Refined quantum entropy inequalities

Quantum local recoverability $[\beta_{\mathrm{QLR}}]$



- Classical $I(A:B|C)_P = \inf_{Q_{B|C}} D_{\mathrm{KL}}\left(P_{ABC} || Q_{B|C} P_{AC}\right)$ $[\beta_{\mathrm{LR}}]$
- Recoverability notion via quantum channels $R_{C\to BC}$ gives:

a. Equality condition $I(A:B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = R_{C\to BC}(\rho_{AC})$ [Petz CMP 1986]

b. No-go $I(A:B|C)_\rho \ngeq \inf_{R_{C\to BC}} D(\rho_{ABC} || R_{C\to BC}(\rho_{AC}))$ [Winter & Li 2008] [Fawzi² JPA 2018]

c. Theorem: $I(A:B|C)_\rho \geq D_{\mathrm{ALL}}\left(\rho_{ABC} || R^{\beta_0}_{C\to BC}(\rho_{AC})\right) \geq \inf_{R_{C\to BC}} D_{\mathrm{ALL}}(\rho_{ABC} || R_{C\to BC}(\rho_{AC}))$

$[\beta_{\mathrm{QLR}}]$

for quantum channel $\mathrm{R}^{\beta_0}_{\mathrm{C}\to BC}(\cdot) := \int_{-\infty}^{\infty} \mathrm{d}\beta_0(t)\, \rho_{BC}^{\frac{1+it}{2}} \rho_C^{\frac{-1+it}{2}} (\cdot) \rho_C^{\frac{-1-it}{2}} \rho_{BC}^{\frac{1-it}{2}}$

[Fawzi & Renner, CMP 2015] [Wilde PRSA 2015] [Junge *et al.*, AHP 2018] [Sutter, **B.**, Tomamichel, CMP 2017] [**B.**, Hirche, Brandão CMP 2021]

Quantum Markov chains?

Quantum Markov chains $[\beta_{\mathrm{QMC}}]$

- Classical $I(A:B|C)_P = \inf_{Q_{A-C-B}} D_{\mathrm{KL}}(P_{ABC}||Q_{A-C-B})$ $[\beta_{\mathrm{MC}}]$ to quantum equality condition

$I(A:B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = \bigoplus_k p_k\ \rho^k_{AC^k_L} \otimes \rho^k_{C^k_R B}$, $C = \bigoplus_k C^k_L \otimes C^k_R$ [Hayden *et al.* CMP 2004]

- No-go $I(A:B|C)_\rho \ngeq \inf_{\sigma \in \mathrm{QMC}(A-C-B)} D_{\mathrm{ALL}}(\rho_{ABC}||\sigma_{ABC})$ [Ibinson, Linden, Winter CMP 2008]

- Separability no-go $I(A:B|C)_\rho \ngeq \inf_{\sigma \in \mathrm{SEP}(A:B)} D_{\mathrm{ALL}}(\rho_{AB}||\sigma_{AB})$ [Christandl, Schuch, Winter CMP 2012]

- Theorem: $I(A:B|C)_\rho \geq \inf_{\sigma \in \mathrm{SEP}(A:B)} D_{\mathrm{LOCC}_1(B\to A)}(\rho_{AB}||\sigma_{AB}) \gtrsim |A|^{-2} \cdot \inf_{\sigma \in \mathrm{SEP}(A:B)} ||\rho_{AB} - \sigma_{AB}||_1^2$

[Brandão, Christandl, Yard, CMP 2011] [Li & Winter CMP 2014] [**B.**, Tomamichel CMP 2024] $[\beta_{\mathrm{QSEP}}]$

- Conjecture: $I(A:B|C)_\rho \geq \inf_{\sigma \in \mathrm{QMC}(A-C-B)} D_{\mathrm{LO}(A:B:C)}(\rho_{ABC}||\sigma_{ABC})$ $[\beta_{\mathrm{QMC}}]$?

cf. latest progress [Salzmann, Bergh, Datta 2024]

Tools in action: Proof of $[\beta_{\mathbf{QSEP}}]$

Refined quantum inequality $[\beta_{\mathrm{QSEP}}]$

- Theorem: $I(A{:}B|C)_\rho \geq E_{\mathrm{LOCC}_1(B\to A)}(A{:}B) - \{E(A{:}C) - E_{\mathrm{ALL}}(A{:}C)\}$ $[\beta'_{\mathrm{QSEP}}]$

with shorthand notation for relative entropies of entanglement

$$E(A{:}B) \coloneqq \inf_{\sigma\in\mathrm{SEP}(A:B)} D(\rho_{AB}||\sigma_{AB}),\ E_{\mathrm{M}}(A{:}B) \coloneqq \inf_{\sigma\in\mathrm{SEP}(A:B)} D_{\mathrm{M}}(\rho_{AB}||\sigma_{AB})$$

- Corollary: $I(A{:}B|C)_\rho \geq E^{\infty}_{\mathrm{LOCC}_1(B\to A)}(A{:}B) \geq E_{\mathrm{LOCC}_1(B\to A)}(A{:}B)$ $[\beta_{\mathrm{QSEP}}]$

via $[\beta'_{\mathrm{QSEP}}]$ for $\rho^{\otimes n}$ and version of asymptotic achievability [Hiai & Petz CMP 1991]

$$D^{\infty}_{\mathrm{ALL}}(\rho||\sigma) \coloneqq \lim_{n\to\infty} \frac{1}{n} D_{\mathrm{ALL}}\left(\rho^{\otimes n}||\sigma^{\otimes n}\right) = D(\rho||\sigma)$$

- Bonus: Optimizer $\sigma_{A^n:B^n} = \int_{-\infty}^{\infty} \mathrm{d}\beta_0(t) \mathrm{Tr}_{C^n}\left[\left(\rho_{BC}^{(1+it)/2}\rho_C^{(-1+it)/2}\right)^{\otimes n}\sigma_{A^n:C^n}\left(\rho_C^{(-1-it)/2}\rho_{BC}^{(1\pm it)/2}\right)^{\otimes n}\right]$

Proof $[\beta'_{\mathrm{SEP}}]$: Variational formulae

- Claim: $I(A:B|C)_\rho + E(A:C)_\rho \geq E_{\mathrm{LOCC}_1(B\to A)}(A:B) + E_{\mathrm{ALL}}(A:C)$
- Proof via $I(A:B|C)_\rho + E(A:C)_\rho = D(\rho_{ABC} || \exp(\log \rho_{BC} + \log \sigma_{A:C} - \log \rho_C)) =: (*)$

  for $\sigma_{A:C} := \mathrm{argmin}_{\sigma\in \mathrm{SEP}(A:C)} D(\rho_{AC}||\sigma_{AC}) \equiv \sum_j \sigma_A^j \otimes \sigma_C^j$

- Variational formula for quantum relative entropy

$$(*) = \sup_{\omega\geq 0} \mathrm{Tr}[\rho_{ABC} \log \omega_{ABC}] - \log \mathrm{Tr}[\exp(\log \omega_{ABC} + \log \rho_{BC} + \log \sigma_{A:C} - \log \rho_C))]$$

  and with choice $\omega_{ABC} = \exp(\log X_{B\to A} + \log Y_{AC}) \geq 0$ for $X_{B\to A} \in \mathrm{LOCC}_1(B \to A)$

$$(*) \geq \sup_{X\in \mathrm{LOCC}_1(B\to A), Y\geq 0} \mathrm{Tr}[\rho_{AB} \log X_{B\to A}] + \mathrm{Tr}[\rho_{AC} \log Y_{AC}] - \log \mathrm{Tr}[\exp(5 \text{ summands})]$$

Proof $[\beta'_{\mathrm{SEP}}]$: Multivariate trace inequality

- 5-matrix Golden-Thompson inequality for $p = 2$ leads to further lower bound

$$(*) \geq \sup_{X \in \mathrm{LOCC}_1(B \to A), Y \geq 0} \mathrm{Tr}[\rho_{AB} \log X_{B\to A}] + \mathrm{Tr}[\rho_{AC} \log Y_{AC}] - \log \mathrm{Tr}\left[Y_{AC} \int_{-\infty}^{\infty} \mathrm{d}\beta_0(t)\, X_{B\to A}^{\frac{1+it}{2}} \gamma_{A:BC}^t X_{B\to A}^{\frac{1-it}{2}}\right] =: (+)$$

for $\gamma_{A:BC}^t \coloneqq \sum_j \sigma_A^j \otimes \left(\rho_{BC}^{(1+it)/2} \rho_C^{(-1-it)/2} \sigma_C^j \rho_C^{(-1+it)/2} \rho_{BC}^{(1-it)/2}\right) \in \mathrm{SEP}(A{:}BC)$

- Defining the state $\gamma_{A:BC} \coloneqq \int_{-\infty}^{\infty} d\beta_0\,(t)\, \gamma_{A:BC}^t$ this can be written as

$$(+) = \sup_{X \in \mathrm{LOCC}_1(B \to A), Y \geq 0} \mathrm{Tr}[\rho_{AB} \log \mathrm{X}_{B\to A}] + \mathrm{Tr}[\rho_{AC} \log Y_{AC}] - \log \mathrm{Tr}[X_{B\to A} \gamma_{A:B}] \cdot \mathrm{Tr}[Y_{AC} \hat{\gamma}_{A:C}]$$

and if $\hat{\gamma}_{A:C} \coloneqq \dfrac{\mathrm{Tr}_B\left[\int_{-\infty}^{\infty} \mathrm{d}\beta_0\,(t) X_{B\to A}^{(1+it)/2} \gamma_{A:BC}^t X_{B\to A}^{(1-it)/2}\right]}{\mathrm{Tr}[X_{B\to A} \gamma_{A:B}]} \in \mathrm{SEP}(A{:}C)$ this would give the sought-after

lower bound $E_{\mathrm{LOCC}_1(B\to A)}(A{:}B)_\rho + E_{\mathrm{ALL}}(A{:}C)_\rho$

Proof $[\beta'_{\mathrm{SEP}}]$: One-way LOCC property

- Altogether $I(A:B|C)_\rho + E(A:C)_\rho \geq E_{\mathrm{LOCC}_1(B\to A)}(A:B)_\rho + E_{\mathrm{ALL}}(A:C)_\rho$
- Missing justification last step:
  - To show $\hat{\gamma}_{A:C} := \frac{\mathrm{Tr}_B\left[\int_{-\infty}^{\infty} \mathrm{d}\beta_0\,(t) X_{B\to A}^{(1+it)/2} \gamma_{A:BC}^{t} X_{B\to A}^{(1-it)/2}\right]}{\mathrm{Tr}[X_{B\to A}\gamma_{A:B}]} \in \mathrm{SEP}(A:C)$ it is crucial that $X_{B\to A} \in \mathrm{LOCC}_1\,(B \to A)$ and thus $E_{\mathrm{LOCC}_1(B\to A)}(A:B)_\rho$ must appear in lower bound
  - Follows as wlog $X_{B\to A} = \sum_x \omega_A^x \otimes P_B^x$ with $\{P_B^x\}$ mutually orthogonal projectors

qed

- Sole inequalities used are

i. *interaction Hamiltonian* $\omega_{ABC} = \exp(\log X_{B\to A} + \log Y_{AC})$ with $X_{B\to A} \in \mathrm{LOCC}_1\,(B \to A)$

ii. 5-matrix Golden-Thompson

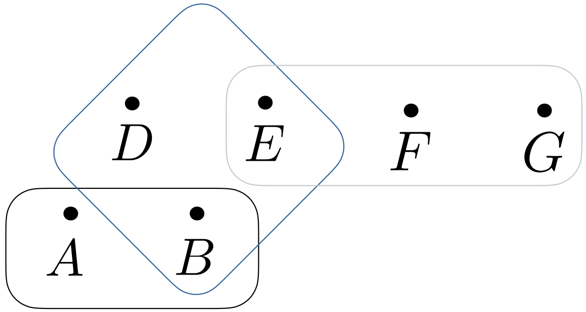
Bonus: More linear entropy inequalities

Non-Shannon inequalities $[\gamma]$

- More linear entropy inequalities, aka entropy cone, e.g., [Zhang & Yeung IEEE 1998]

$$I(A:B)_P + I(A:CD)_P + 3 \cdot I(C:D|A)_P + I(C:D|B)_P \geq 2 \cdot I(C:D)_P \; [\gamma_{\text{NS}}]$$

→ in fact, infinitely many more such inequalities
[Matus IEEE 2007]



- Markov chain refinement $[\beta_{\text{MC}}]$ of standard $[\alpha_{\text{SSA}}]$ crucial step in derivation

$$I(A:B|C)_P = \inf_{Q_{A-C-B}} D_{\text{KL}}(P_{ABC} || Q_{A-C-B})$$

- Conjecture: Quantum version $[\gamma_{\text{QNS}}]$ again due to monogamy of entanglement?

[Linden & Winter CMP 2005] [Majenz 2018]

Conclusion

Summary

- From classical to quantum entropy inequalities:
  - Standard inequalities $[\alpha]$ ✅
  - Local recoverability refinement $[\beta_{\mathrm{QLR}}]$ ✅
  - Markov chain refinement in terms of separability $[\beta_{\mathrm{SEP}}]$ ✅
  - Markov chain refinement $[\beta_{\mathrm{QMC}}]$ ❓
  - Non-Shannon type inequalities $[\gamma_{\mathrm{QNS}}]$ ❓

→ cascading steps

- Matrix analysis directly leads to tight and most general bounds, enablers:

A. Multivariate matrix trace inequalities

B. Variational formulae for different quantum relative entropies, including, e.g., (locally) measured ones

Open questions

- Quantum Markov chain and non-Shannon type conjectures
- Find more multivariate trace inequalities, e.g., logarithmic ones

  → Belavkin-Staszewski type tight upper bounds? cf. related work [Bluhm *et al.* 2025]
- Find more multipartite extensions [**B.** & Tomamichel CMP 2024]
- Bonus: Tight entropic uncertainty relations for multiple measurements

  [Frank & Lieb CMP 2013] [**B.**, Sutter, Walter CMP 2023]

- Many more, with applications throughout science…

  - Open postdoc and PhD positions at RWTH Aachen University
  - Get in contact:

    berta@physik.rwth-aachen.de

RWTH AACHEN UNIVERSITY

Thank you.

Extra material

Research group

- Institute for Quantum Information (IQI)
- Theory of quantum information science
- Members:

JARA INSTITUTE QUANTUM INFORMATION

AN INITIATIVE OF

RWTH AACHEN UNIVERSITY

JÜLICH Forschungszentrum

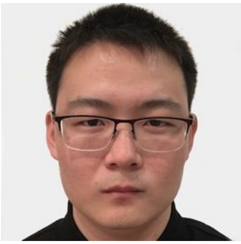
Mario Berta
Professor of Physics

Sreejith Sreekumar
Postdoc

Aditya Nema
Postdoc

Aadil Oufkir
Postdoc

Yongsheng Yao
Postdoc



Michael X. Cao
Postdoc

+ open PhD and postdoc positions, get in contact!

Tobias Rippchen
PhD student

Julius Zeiss
PhD student

Gereon Koßmann
PhD student

Nikolaos Louloudis
PhD student

Richard Meister
Postdoc (Imperial)

+ 5x Master students RWTH Physics / Computer Science

Theory of quantum information science

- Our focus areas:

  I. Mathematical foundations of quantum information
  II. Quantum algorithm development

- Cluster of Excellence: Matter and Light for Quantum Computing (ML4Q) [renewed!]
- Visiting Reader at Department of Computing Imperial College London
- Industry ties with Amazon Web Services Center for Quantum Computing