Asymptotic quantification of entanglement with a single copy

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Introduction. A natural way of benchmarking quantum information processing protocols is in terms of their asymptotic rates: they quantify the ultimate performance that can be reached when many copies of a given state or channel are available. In the study of quantum entanglement — which is the main focus of this work — such rates provide a precise way of quantifying the entanglement content of a given state, directly relating it with the practical performance of some operational task such as entanglement distillation or dilution.

However, unlike classical information theory, the description of the asymptotic rates in quantum information typically cannot be done using simple entropic quantities. Even when one can connect the performance of a given task with a closed-form quantity — such as, for example, the quantum relative entropy [1], the coherent information of a channel [2–4], or the entanglement of formation [5, 6] — the ubiquity of non-additivity phenomena in quantum information [7–11] means that the optimal asymptotic rates can only be expressed using *regularised* quantities of the form $\lim_{n\to\infty} \frac{1}{n} f(\rho_{AB}^{\otimes n})$. These are immensely difficult to evaluate even for simple functions f, preventing an efficient quantitative characterisation of the ultimate limits of quantum information processing, even under simplified assumptions or relaxed constraints.

This motivates us to question the considered setting itself. **If precise answers are so hard to find, maybe we are not asking the right questions?** Could we obtain a simpler characterisation of asymptotic entanglement manipulation tasks by changing the way that we benchmark the performance of the protocols, shifting the focus to another figure of merit?

Summary of main results. Inspired by the information-theoretic characterisation of quantum state discrimination [12–14], where one is concerned with the rate of decay of the error, we adopt the same approach in the description of the fundamental protocol of quantum entanglement distillation. This leads us to a **fundamentally new paradigm to quantify entanglement in the asymptotic limit**. Previous studies focused on the *yield* of this process, i.e. the maximum amount of pure entanglement that can be extracted with the constraint that the error vanishes asymptotically; this yield, known as distillable entanglement, turns out to be exceedingly difficult to compute for even the simplest quantum systems, and it is therefore not practically viable as an entanglement measure. The conceptual shift we propose is to focus instead on the *quality* (rather than the quantity) of the obtained entanglement, represented by the optimal rate of decay of the distillation error, while still guaranteeing that the protocols output an arbitrarily large amount of pure entanglement asymptotically.

This error scaling constitutes a new asymptotic measure of entanglement that is operationally meaningful, since it *exactly* quantifies the performance of an important asymptotic entanglement protocol. Remarkably, as we show here, it is the **very first** such measure that is **single letter**, meaning that it is computable with just a single copy of a given state and no regularisation is needed. This contrasts with known operational measures of entanglement, showing that the task of benchmarking asymptotic entanglement protocols may not always be as difficult as previously thought. Our results rely on two main contributions that we make in this work.

1. **Connection with entanglement testing**. We first establish an exact equivalence between evaluating the error of entanglement distillation under non-entangling operations and a composite quantum hypothesis testing task known as entanglement testing, extending the relation previously found in a related setting by Brandão and Plenio [15]. This connection gives us a different way of looking at the problem, and in particular allows us to use information-theoretic techniques to tackle the task of entanglement distillation. Indeed, computing the asymptotic rate of error scaling in entanglement testing is a generalisation of a result in quantum hypothesis testing known as quantum Sanov's theorem [16–18]. However, the much more complicated structure involved in the problem we encounter here means that no known results are sufficiently general to shed any light on it.

2. **Full solution of asymptotic entanglement testing.** As our main contribution, we then evaluate exactly the asymptotic performance of entanglement distillation and establish an exact expression for its asymptotic error exponent by solving the generalised quantum Sanov's theorem in full generality. We in particular show that the exponent is given by a variant of the relative entropy of entanglement, the ***reverse relative entropy*** $D(\mathcal{S}_{A:B}\|\rho_{AB})$ [1, 19]. The remarkable aspect of this result is that the quantity can be evaluated exactly — without regularisation — on a single copy of the given quantum state, circumventing the problems that affect other measures of entanglement connected with practical tasks. Our result thus establishes the reverse relative entropy as a measure of entanglement with a direct operational meaning in entanglement distillation, while at the same time being computable without having to evaluate a many-copy limit.

The main significance of our results is the demonstration that truly asymptotic properties of entanglement can be characterised exactly without the need to consider asymptotic, regularised entanglement measures. This is important from a practical perspective — as evaluating regularised quantities is typically extremely hard, making it difficult to quantify optimal rates and give benchmarks on feasible protocols — but also from a theoretical one, as single-letter expressions are much easier to characterise mathematically and can lead to an improved theoretical understanding of the ultimate limitations of entanglement manipulation. Previously, such single-letter results were known only in very limited special cases, e.g. pure [20] or maximally correlated states [21]. To the best of our knowledge, our work represents the first solution of an asymptotic entanglement transformation protocol, in the sense of an asymptotic task with vanishing error, that admits a single-letter solution for all quantum states. A comparison with the generalised quantum Stein's lemma [22–24] is particularly illuminating: while in the Stein setting one obtains the hard-to-compute *regularised* relative entropy of entanglement as the optimal exponent, here the solution is instead single letter.

A conclusion that one may draw from our approach is that it can be beneficial to shift the focus from the asymptotic yield to the error scaling in asymptotic manipulation protocols. This seemingly simple insight opened the door to new developments in our understanding of entanglement manipulation: it allowed us to obtain a complete, single-letter solution of the distillable entanglement error exponent. The basic idea can be immediately generalised to other settings in the study of quantum and classical information, and we hope that this will lead to many more fruitful connections and developments in quantum information processing.

Setting of entanglement distillation. Alice and Bob aim to apply some protocol $(\Lambda_n)_n$ on n copies of their shared state ρ_{AB} such that the final state approximates m copies of the two-qubit maximally entangled state $|\Phi_+\rangle$, up to an error ε_n. Now, if we understand $\frac{m}{n}$ as the yield of this protocol, the *distillable entanglement* $E_d(\rho_{AB})$ is then defined as the largest asymptotic yield $\lim_{n\to\infty}\frac{m}{n}$ optimised over all feasible protocols such that the transformation is asymptotically perfect, that is, $\lim_{n\to\infty}\varepsilon_n = 0$.

Following the axiomatic framework of Brandão and Plenio [15, 25], we consider here all non-entangling (NE) protocols Λ_n, that is, quantum channels which are unable to generate any entanglement: $\Lambda_n(\sigma)$ must remain separable for all separable σ. This approach, inspired by axiomatic approaches to the second law of thermodynamics [26, 27], naturally leads to a well-behaved framework for entanglement manipulation. It can be regarded as a relaxation of the more traditional framework of local operations and classical communication (LOCC) that avoids the extremely complicated structure of the latter [28].

Now, in the setting of quantum hypothesis testing, one is tasked with distinguishing between two quantum states ρ and σ by performing a collective measurement on n copies of them. The probability of mistaking ρ for σ decays exponentially as 2^{-cn}, and it is this *error exponent* c that one aims to quantify in order to understand how fast the distinguishability improves. Remarkably, in the limit as $n\to\infty$, the quantum Stein's lemma tells us that the exponent exactly equals the quantum relative entropy $D(\sigma\|\rho)$ [12, 13].

We thus apply a similar reasoning to entanglement distillation and ask about the distillation error exponent. Specifically, consider again a distillation protocol that outputs m copies of $|\Phi_+\rangle$ with error ε_n. We will now ask: how fast does the quality of the distilled entanglement improve as the number of distilled copies m grows to infinity? Instead of focusing on the optimal yield m/n when $\varepsilon_n \to 0$, we will thus require that $m\to\infty$ and characterise the optimal error exponent c such that $\varepsilon_n \sim 2^{-cn}$ is achievable. This quantity, called the *distillable entanglement error exponent*,[1] is defined by

$$E_{d,\mathrm{err}}(\rho) := \lim_{m\to\infty}\ \sup\left\{ \lim_{n\to\infty} -\frac{1}{n}\log\varepsilon_n \ \middle|\ \Lambda_n(\rho_{AB}^{\otimes n}) \approx_{\varepsilon_n} |\Phi_+\rangle\langle\Phi_+|^{\otimes m},\ \Lambda_n\in\mathrm{NE}\right\}, \tag{1}$$

where we again optimise over all non-entangling distillation protocols to find the least achievable error. One can notice that this definition no longer places any importance on the precise number of maximally entangled copies that we obtain in the protocol (provided that it can be made as large as desired), but only on the exponentially decreasing error. This provides an alternative angle of looking at the performance of distillation protocols, incomparable with previous approaches that focused on the distillation yield.

Connection with hypothesis testing. The task of entanglement testing is a composite hypothesis testing problem: we are to distinguish between $\rho_{AB}^{\otimes n}$ and the whole set of separable quantum states $\mathcal{S}_{A^n:B^n}$ with a measurement. Just as in conventional hypothesis testing, we would like to understand the behaviour of the error exponent for large n. There is, however, a certain freedom in choosing which type of error we quantify here. The so-called type I error occurs when we mistake $\rho_{AB}^{\otimes n}$ for a separable state, while a type II error occurs when we mistake a separable state for $\rho_{AB}^{\otimes n}$. For a fixed type I error probability, the asymptotic exponent of the type II error probability is known as the *Stein exponent* $\mathrm{Stein}(\rho_{AB}\|\mathcal{S}_{A:B})$; conversely, the exponent of the type I error probability is known as the *Sanov exponent*. We will denote the latter by $\mathrm{Sanov}(\rho_{AB}\|\mathcal{S}_{A:B})$. The Stein exponent of entanglement testing was first investigated in the works of Brandão and Plenio [15], where it was connected with the asymptotic yield of entanglement distillation. Here we establish a dual to that result, proving an exact connection between the Sanov exponent and the error of entanglement distillation.

Lemma 1. *The asymptotic error exponent of entanglement distillation under non-entangling operations equals the Sanov error exponent of hypothesis testing of all separable states $\mathcal{S}_{A:B}$ against ρ_{AB}:*

$$E_{d,\mathrm{err}}(\rho_{AB}) = \mathrm{Sanov}(\rho_{AB}\|\mathcal{S}_{A:B}). \tag{2}$$

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[1] We note here that the term *error exponent* is often used to mean a different quantity in information theory, namely, the rate of decay of error for a *fixed* yield rate. Our definition asks about the optimal rate of error decay regardless of the final yield.

A generalised quantum Sanov's theorem. We now define the *reverse relative entropy of entanglement* as [19]

$$D(\mathcal{S}_{A:B}\|\rho_{AB}) := \min_{\sigma_{AB}\in\mathcal{S}_{A:B}} D(\sigma_{AB}\|\rho_{AB}), \tag{3}$$

where the term 'reverse' refers to the fact that the relative entropy of entanglement was originally defined with the arguments in the opposite order, as $D(\rho_{AB}\|\mathcal{S}_{A:B}) := \min_{\sigma_{AB}\in\mathcal{S}_{A:B}} D(\rho_{AB}\|\sigma_{AB})$ [1].

Our main result is the complete solution of the Sanov exponent of entanglement testing, which by Lemma 1 also gives a resolution of the error exponent of entanglement distillation.

Theorem 2. *For any state ρ_{AB}, the asymptotic Sanov error exponent of entanglement testing equals the reverse relative entropy of entanglement:*

$$E_{d,\mathrm{err}}(\rho_{AB}) = \mathrm{Sanov}(\rho_{AB}\|\mathcal{S}_{A:B}) = D(\mathcal{S}_{A:B}\|\rho_{AB}). \tag{4}$$

A remarkable aspect here is that, although both the distillable entanglement error exponent $E_{d,\mathrm{err}}$ and the Sanov exponent express *asymptotic* operational properties of the state — they quantify the performance of $\rho_{AB}^{\otimes n}$ in the limit of large n — the quantity $D(\mathcal{S}_{A:B}\|\rho_{AB})$ can be evaluated on a single copy of ρ_{AB}. This lets us avoid the biggest issue that plagues most solutions of asymptotic rates of entanglement manipulation protocols.

Technical aspects. The main technical hurdle in proving Theorem 2 is that the hypotheses (states) involved in the discrimination task depart from the typically considered setting of independent and identically distributed (i.i.d.) ones. The recent years have seen significant interest in such hypothesis testing tasks 'beyond i.i.d.' [16–18, 22–24, 29–34], but none of the previous results are sufficiently general to cover the case of entanglement testing. In particular, Hayashi and Ito [18] recently obtained bounds on the Sanov exponent of entanglement testing in terms of a regularisation of the Petz–Rényi divergences of entanglement $D_\alpha(\mathcal{S}_{A:B}\|\rho_{AB})$. There, the question was raised whether these diverges could be additive, in which case an exact expression for the asymptotic exponent could be obtained. However, we show that this additivity does *not* hold: we find that $D_\alpha(\mathcal{S}_{A:B}\|\rho_a\otimes\rho_a) < 2\,D_\alpha(\mathcal{S}_{A:B}\|\rho_a)$ for the antisymmetric Werner state ρ_a. This means that it is impossible to circumvent the need for regularisation in the results of [18], preventing us from using the known continuity results for the Petz–Rényi divergences and following the approach of Hayashi and Ito to compute the exponent exactly. We thus need to develop a novel approach to tackle this problem.

Our proof of the Theorem proceeds in two steps. First, we prove the result in the case of classical information theory. Despite the mathematically simpler structure, this is already non-trivial as the intuitive approaches used in i.i.d. cases — such as the aforementioned Petz–Rényi divergences — fail to work. Instead, we employ a recently introduced powerful mathematical technique called blurring [24]. This technique was developed in the context of the generalised quantum Stein's lemma [22], which is a result complementary to Sanov's theorem — the previous results of [24] are very different and do not suffice to evaluate the Sanov exponent. However, through a careful combination with fundamental information-theoretic results in the theory of types [35, 36], we are able to use blurring to handle general composite problems in hypothesis testing in the Sanov setting.

To complete our solution, the classical result needs to be lifted to a fully quantum one. We accomplish this by measuring the quantum systems and employing a technically delicate reduction procedure. Our solution of the problem in fact extends beyond entanglement testing to the testing of more general quantum resources.

Comparison and discussion. Our result may be compared with previous findings that connected rates of entanglement distillation with hypothesis testing problems. This notably includes the generalised quantum Stein's lemma, conjectured in [22] and recently proven in [23, 24], which states that the Stein exponent of entanglement testing is given by the regularised relative entropy of entanglement $D^\infty(\rho_{AB}\|\mathcal{S}_{A:B}) := \lim_{n\to\infty}\frac{1}{n}D(\rho_{AB}^{\otimes n}\|\mathcal{S}_{A:B})$. By [15], this equals the distillable entanglement yield under NE operations. In addition to the different proof methods, the major difference between this result and ours is the need for regularisation: although the generalised Stein's lemma ostensibly provides an exact expression of the distillable entanglement, it is given by a regularised quantity, which prevents an efficient evaluation of it except for limited special cases. Our Theorem 2 does not suffer from such a hindrance, as it establishes a single-letter formula.

We note that there are instances of quantum states from which maximal entanglement can be distilled exactly, with no error — e.g., pure entangled states $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ [37]. In such cases the error exponent can be chosen to be arbitrarily high, and so $E_{d,\mathrm{err}}$ diverges to infinity. This is indeed expected and is fully captured by Theorem 2: for all such states, $D(\mathcal{S}_{A:B}\|\rho_{AB}) = \infty$. Although this looks like it may limit the applicability of our result, such cases highly contrast with noisy quantum states encountered in practice: zero-error entanglement extraction is impossible from all full- or high-rank quantum states [38, 39], meaning that $D(\mathcal{S}_{A:B}\|\rho_{AB}) < \infty$ for generic ρ_{AB}. It is precisely these noisy states for which computing the asymptotic rates of entanglement distillation has been such a difficult task, as conventional techniques in entanglement distillation, which enabled a complete description of distillation for pure states [20], did not manage to shed much light on the general case of mixed states. This means that our results can find direct practical applicability as an entanglement benchmark in the regime complementary to the well-studied and well-understood setting of noiseless pure states, serving as an operational entangled measure for the most practically relevant noisy quantum systems.

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