Optimality of meta-converse for channel simulation

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Channel coding

Noisy channel coding

- One-shot task: given channel $W_{X\to Y}(y|x)$ and message size M, compute success probability

$$\mathrm{p_{cod}}(W,M) := \sup_{(E,D)} \frac{1}{M} \sum_{x,y,i} W_{X\to Y}(y|x) E(x|i) D(i|y)$$

over encoder-decoder pairs

- Shannon theorem: largest R with $\mathrm{p_{cod}}(W^{\times n}, R^n) \to 1$ in limit $n \to \infty$ is quantified by channel capacity $C(W)$ with mutual information formula

$$C(W) = \sup_{P_X} I(X\colon Y)_{W(P)}$$

- Finite-blocklength refinements?

Meta converse for channel coding

- Bottom-up approach for Shannon theory, natural meta converse linear program relaxation

$$\mathrm{p}_{\mathrm{cod}}(\mathrm{W},\mathrm{N}) \leq \mathrm{p}_{\mathrm{cod}}^{\mathrm{NS}}(W,M) := \sup_{(r,p)} \frac{1}{M}\sum_{x,y} W_{X\to Y}(y|x)r(x,y)$$

with $\sum_{x,y} r(x,y) \leq 1, \sum_x p_x = M, r(x,y) \leq p_x, 0 \leq r(x,y), p(x) \leq 1$

- Great analytical properties for $W_{X\to Y}^{\times n}$ asymptotic $n \to \infty$ expansion
- Tight up to third-order + for strong converse exponent, but not for error exponent (no critical rate!)
- Physics: exactly corresponds to non-signaling assisted value

Non-signaling assistance

- $p_{\text{cod}}^{\text{NS}}(W, M)$ allows for joint encoder-decoder maps $P(x, j|i, y) \geq 0$ with
  $$\sum_{x,j} P(x, j|i, y) = 1 \;\forall\; i, y$$
  $$\sum_{j} P(x, j|i, y) = P_A(x|i) \;\forall\; x, i, y \;\;\&\;\; \sum_{x} P(x, j|i, y) = P_B(j|y) \;\forall\; j, i, y$$
  for some $P_A(x|i)$ and $P_B(j|y)$
- Corresponds to any physical correlations that do not allow for signaling (=instantaneous transformation of information in local hidden variable models)
- Includes, e.g., quantum (=entanglement) or shared randomness assistance

Optimality of meta converse for channel coding

- Bottom-up approach to Shannon theory: one-shot optimality of meta converse?
- Yes, rounding methods from approximation algorithms give for M, N that

$$p_{\mathrm{cod}}^{\mathrm{NS}}(W,N) \geq \mathrm{p}_{\mathrm{cod}}(W,N) \geq \frac{M}{N} \cdot \left(1-\left(1-\frac{1}{M}\right)^{N}\right) \cdot p_{\mathrm{cod}}^{\mathrm{NS}}(W,M)$$

which implies the constant factor approximation

$$p_{\mathrm{cod}}^{\mathrm{NS}}(W,M) \geq p_{\mathrm{cod}}(W,M) \geq \frac{1}{(1-e^{-1})} \cdot p_{\mathrm{cod}}^{\mathrm{NS}}(W,M)$$

- Bound is tight + gives strong upper bound on entanglement assistance (NB: also correct strong converse exponent – to appear)

Channel interconversion

Channel interconversion

- One-shot task: given channels $W_{X\to Y}(y|x)$ and $\mathrm{V}_{\mathrm{X}\to Y}(y|x)$, at what rate can one channel simulate the other one?
- Reverse Shannon theorem: with shared randomness assistance the capacity becomes

$$C(W \to V) = \frac{C(W)}{C(V)}$$

  with respective mutual information formulas → beautiful!
- (Finite) shared randomness needed for reversibility, but finite-blocklength refinements unknown
- Look at simpler, reverse task of channel coding: channel simulation

Channel simulation

Channel simulation

- One-shot task: given channel $V_{X\to Y}(y|x)$ and identity channel of size M compute, compute shared randomness assisted success probability

$$\mathrm{p}_{\mathrm{sim}}^{\mathrm{SR}}(V,M) := \sup_{(p_{\mathrm{SR}},E,D)} 1 - \sup_{x} ||\tilde{V}_{X\to Y}(\cdot\,|x) - V_{X\to Y}(\cdot\,|x)||_{\mathrm{TV}}$$

  over synthesized channels $\tilde{V}_{X\to Y}(y|x) := \sum_s p_{\mathrm{SR}}(s) \sum_i E_s(i|x) D_s(y|i)$

  with randomness assisted encoder-decoder pairs
- Reverse Shannon theorem: smallest R with $\mathrm{p}_{\mathrm{sim}}^{\mathrm{SR}}(V^{\times n}, R^n) \to 1$ in limit $n \to \infty$ is quantified by channel capacity

$$C(V) = \sup_{P_X} I(X{:}Y)_{V(P)}$$

Meta converse for channel simulation

- Bottom-up approach for Shannon theory, natural meta converse linear program relaxation

$$\mathrm{p}_{\mathrm{sim}}^{\mathrm{SR}}(V,M) \leq p_{\mathrm{sim}}^{\mathrm{NS}}(V,M) := \sup_{(U,q)} 1 - \sup_{x} ||U_{X\to Y}(\cdot\,|x) - V_{X\to Y}(\cdot\,|x)||_{\mathrm{TV}}$$

over channels $U_{X\to Y}(y|x)$ with $U_{X\to Y}(y|x) \leq q(y)$ and $\sum_y q(y) = M$

- Great analytical properties for $V_{X\to Y}^{\times n}$ asymptotic $n\to\infty$ expansion
- Tight up to third-order (only second-order expansion known!) + error exponent & strong converse exponent?
- Physics: exactly corresponds to non-signaling assisted value

Main result

Result: one-shot simulation rounding

- Bottom-up approach to Shannon theory: one-shot optimality of meta converse?
- Yes, rounding methods from approximation algorithms give for M, N that

$$p_{\mathrm{sim}}^{\mathrm{NS}}(V,N) \geq \mathrm{p}_{\mathrm{sim}}^{\mathrm{SR}}(V,N) \geq 1 \cdot \left(1 - \left(1 - \tfrac{1}{M}\right)^{N}\right) \cdot p_{\mathrm{sim}}^{\mathrm{NS}}(V,M)$$

which implies the constant factor approximation

$$p_{\mathrm{sim}}^{\mathrm{NS}}(V,M) \geq p_{\mathrm{sim}}^{\mathrm{SR}}(V,M) \geq \frac{1}{(1-e^{-1})} \cdot p_{\mathrm{sim}}^{\mathrm{NS}}(V,M)$$

- Bound is tight + gives strong upper bound on entanglement assistance
- Follow-up work: strong converse and error exponent (no critical rate!)

Proof ideas

Tightness of gap proof

- The family of channels

$$\mathrm{U}^{(n,k)} : \binom{n}{k} \to \{1,2,\cdots,n\} \text{ with } U_{X\to Y}(y|x) := \frac{1}{k}\, 1\{y \in x\}$$

has for $n = M^2$ and $k = M$ the limit

$$\lim_{M\to\infty} \frac{p_{\mathrm{sim}}^{\mathrm{SR}}\left(U^{(M^2,M)}, M\right)}{p_{\mathrm{sim}}^{\mathrm{NS}}\left(U^{(M^2,M)}, M\right)} = 1 - \mathrm{e}^{-1}$$

which then exactly matches

$$p_{\mathrm{sim}}^{\mathrm{NS}}(V,M) \geq p_{\mathrm{sim}}^{\mathrm{SR}}(V,M) \geq \frac{1}{(1-e^{-1})} \cdot p_{\mathrm{sim}}^{\mathrm{NS}}(V,M)$$

- Crucial step: upper bound power of shared randomness assistance

Rounding proof

- Any feasible solution of linear program $p_{\mathrm{sim}}^{\mathrm{NS}}(V,M)$ gives channel $U_{X\to Y}(y|x)$ and distribution $\frac{1}{M}q(y)$, then

  → construct shared randomness assisted synthesized channel $\tilde{V}_{X\to Y}$ for N such that

$$U_{X\to Y}(y|x) \geq \left(1-\left(1-\tfrac{1}{M}\right)^N\right)\tilde{V}_{X\to Y}(y|x)\ \forall\, x,y$$

- This meta inequality also works for average error / fidelity criteria!
- Basic idea:
  - Shared randomness assistance $\{\frac{1}{M}q(y)\}_y$
  - Rejection sampling with N steps and $t_{\mathrm{initial}}(y)=\frac{1}{M}q(y)$, $t_{\mathrm{target}}(y)=U_{X\to Y}(y|x)$

Quantum extension

Classical-quantum channels

- Classical input – quantum output channels

$$V_{X\to B}: x \mapsto \rho_B^x$$

for sub-normalized quantum states $\rho_B^x \geq 0$, normalized to $\sum_x \mathrm{Tr}[\rho_B^x] = 1$

- Entanglement-assisted simulation protocol

$$\mathrm{p}_{\mathrm{sim}}^{\mathrm{EA}}(V, M) := \sup_{(\sigma,E,D)} 1 - \sup_x ||\tilde{V}_{X\to B} - V_{X\to B}||_1$$

over synthesized channels $\tilde{V}_{X\to B} := \sum_i N_{K\to B}^i \left(\mathrm{Tr}_{K'}\left[\left(E_x^i \otimes 1_{K'}\right)\sigma_{KK'}\right]\right)$

with assistance $\sigma_{KK'}$ + encoders $\{E_x^i\}_i$ and decoders $\{N_{K\to B}^i\}_i$

- Same rounding results – not know for reverse task of channel coding!

Conclusion

- Previous channel coding result

$$p_{\text{cod}}^{\text{NS}}(W,N) \geq \mathrm{p}_{\text{cod}}(W,N) \geq \frac{M}{N} \cdot \left(1 - \left(1 - \frac{1}{M}\right)^{N}\right) \cdot p_{\text{cod}}^{\text{NS}}(W,M)$$

  via maximizing of sub-modular function rounding – tight

- Novel channel simulation result

$$p_{\text{sim}}^{\text{NS}}(V,N) \geq \mathrm{p}_{\text{sim}}^{\text{SR}}(V,N) \geq \left(1 - \left(1 - \frac{1}{M}\right)^{N}\right) \cdot p_{\text{sim}}^{\text{NS}}(V,M)$$

  via rejection sampling rounding – tight

- Channel interconversion? Relaxation via non-signaling assistance

$$p_{\text{sim}}^{\text{NS}}(W \mapsto V) \;\; \text{versus} \;\; p_{\text{sim}}^{\text{SR}}(W \mapsto V)$$

Outlook

- On the arXiv soon
- Follow-up work: strong converse exponent and error exponent of classical channel simulation via linear-program relaxation (no critical rate!)
- Quantum cases: shared randomness vs entanglement assistance vs non-signaling assistance? Hard.
- Back to channel interconversion? Beyond capacity completely open.

  - Postdoc and PhD positions at RWTH Aachen University
  - Get in contact:

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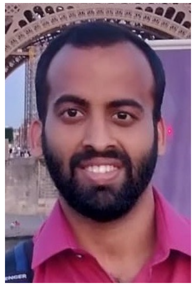
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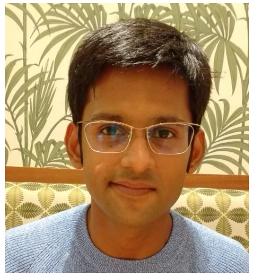
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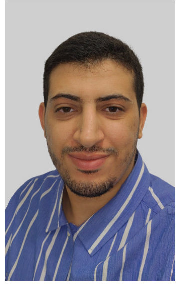
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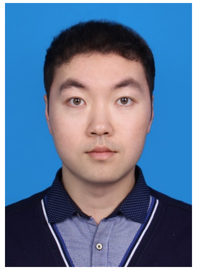
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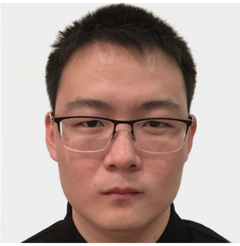
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