From classical to quantum entropy inequalities

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- Theory of quantum information science
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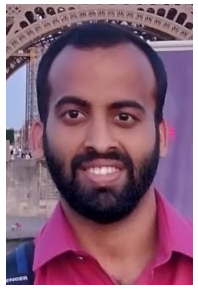

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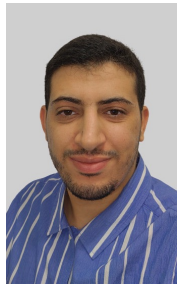

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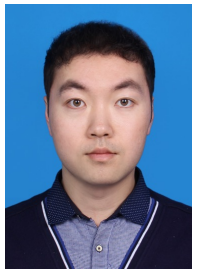

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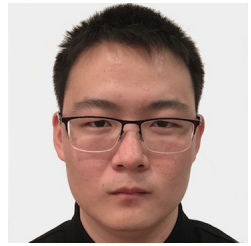

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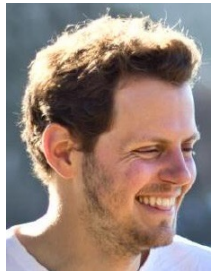

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Theory of quantum information science

- Our focus areas:
  1. Mathematical foundations of quantum information
  2. Quantum algorithm development

- Cluster of Excellence: Matter and Light for Quantum Computing (ML4Q)
- Visiting Reader at Department of Computing Imperial College London
- Industry ties with Amazon Web Services Center for Quantum Computing

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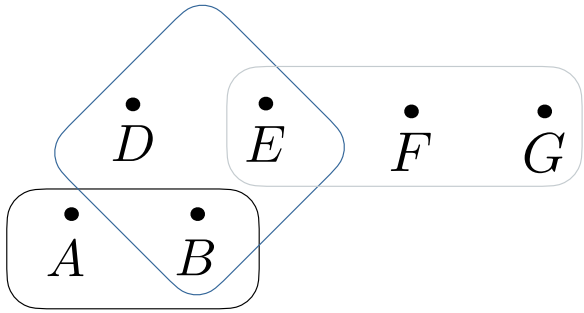
Entropy in the classical world

Entropy function

- Probability distribution P_{ABC} with Shannon entropy:

$$H(A)_P = -\sum_x P_A^x \log P_A^x \geq 0$$



- Mutual information:

$$I(A:B)_P = H(A)_P + H(B)_P - H(AB)_P \geq 0 \text{ (sub-additivity)}$$

- Conditional entropy:

$$H(A|B)_P = H(AB)_P - H(B)_P \geq 0$$

- Conditional mutual information:

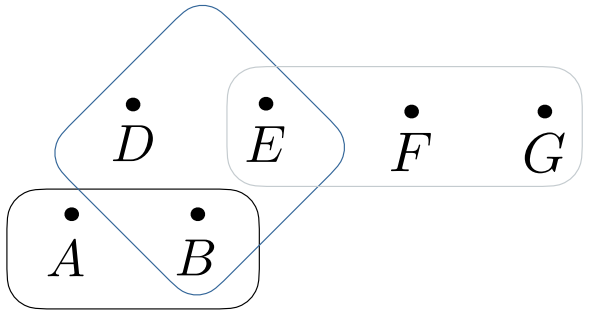
$$I(A:B|C)_P = H(AC)_P + H(BC)_P - H(C)_P - H(ABC)_P \geq 0 \text{ (strong sub-additivity)}$$

More linear entropy inequalities

- Non-Shannon type inequalities (!), aka entropy cone, e.g.,

$$2I(C:D)_P \leq I(A:B)_P + I(A:CD)_P + 3I(C:D|A)_P + I(C:D|B)_P$$

→ in fact, (infinitely) many more



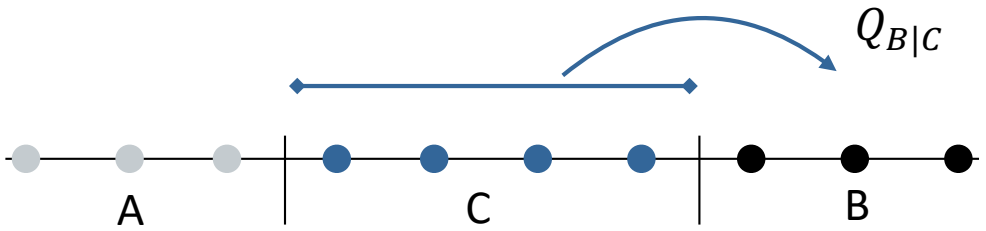
- Parent quantity Kullback-Leibler (KL) divergence

$$D(P||Q) \coloneqq \sum_x P^x \log\left(\frac{P^x}{Q^x}\right)$$

Refinements: strong sub-additivity (SSA)

- Recoverability formulation

$$I(A:B|C)_P = \inf_{Q_{B|C}} D\left(P_{ABC}||Q_{B|C}P_{AC}\right) \geq 0 \text{ with } Q_{BC} = P_{BC}P_C^{-1}$$



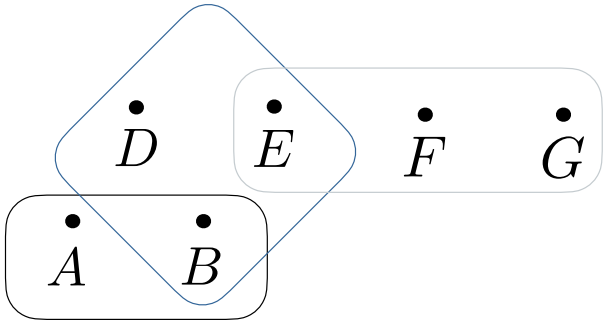
- Markov chain formulation (equivalent)

$$I(A:B|C)_P = \inf_{Q_{A-c-B}} D(P_{ABC}||Q_{A-C-B}) \geq 0 \text{ with } Q_{A-C-B} = P_{BC}P_C^{-1}P_{AC}$$

- Equality conditions $I(A:B|C)_P = 0$ from $D(P||Q) = 0 \Leftrightarrow P = Q$

Quantum information theory

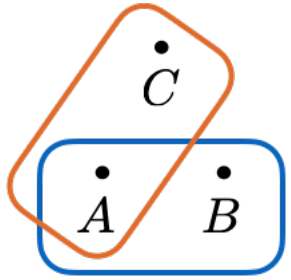
Quantum entropy & entanglement

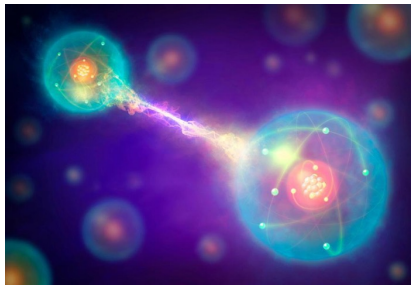


- Classical versus quantum correlations (=entanglement)?

- Example: two maximally quantumly correlated quantum bits (qubits) A and B cannot be quantumly correlated with any C

  → monogamy of entanglement!





- Example: SSA still holds $I(A\colon B|C)_\rho \geq 0$ (highly non-trivial)

- Quantum entropy inequalities ⇔ entanglement structure of quantum states

Quantum formalism

- Quantum systems:
  inner product spaces $\mathcal{H}$ with tensor products $\mathcal{H}_A \otimes \mathcal{H}_B$
- Quantum states:
  $$\mathcal{S}(\mathcal{H}) = \{\rho \in \mathrm{Lin}(\mathcal{H}) |\, \rho \geq 0 \vee \mathrm{Tr}[\rho] = 1\}$$
- Quantum channels:
  $$Q_{ch}(\mathcal{H} \to \mathcal{H}') = \{\mathcal{N}: \mathrm{Lin}(\mathcal{H}) \to \mathrm{Lin}(\mathcal{H}') | \mathcal{N} \text{ compl. pos. \& trace pres.}\}$$
- Example partial trace: $\mathrm{Tr}[Tr_B[X_{AB}]Y_A] = \mathrm{Tr}[X_{AB}(Y_A \otimes 1_B)] \; \forall \; Y_A \in \mathrm{Lin}(\mathcal{H}_A)$

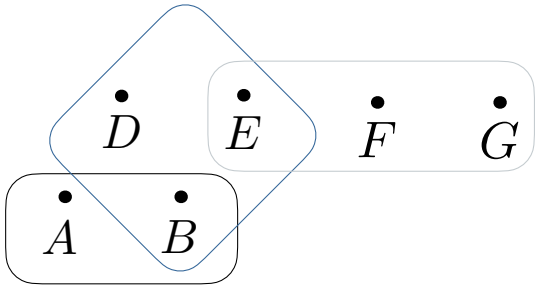
  → write $\mathrm{Tr}_B[\rho_{AB}] = \rho_A$
- Example measurement: $\mathbb{P}_{\rho,M}(x) = \mathrm{Tr}[\rho M^x]$ via $\{M^x \geq 0\}_x$ with $\sum_x M^x = 1$

Quantum entropy

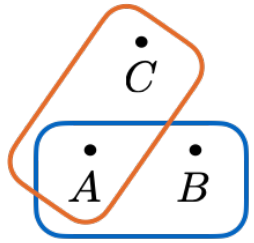
Quantum entropy inequalities

- State $\rho_{ABC} \gneqq 0$, $\mathrm{Tr}[\rho_{ABC}] = 1$ with von Neumann entropy

$$H(A)_\rho = -\mathrm{Tr}[\rho_A \log \rho_A] \geq 0$$



- $I(A\colon B)_\rho = H(A)_\rho + H(B)_\rho - H(AB)_\rho \geq 0$
- $H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$ can get negative!



- Strong subadditivity (SSA) - highly non-trivial:

$$I(A\colon B|C)_\rho = H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho \geq 0$$

- Equivalent to entanglement monogamy = weak monotonicity:

$$H(A|C)_\rho + H(A|B)_\rho \geq 0$$

Parent quantity quantum relative entropy

- Classical KL divergence $D(P||Q) = \sum_x P^x \log\left(\frac{P^x}{Q^x}\right)$
- Quantum relative entropy

$$D(\rho||\sigma) = \mathrm{Tr}[\rho(\log(\rho) - \log(\sigma))]$$ Umegaki

→ simplifies to KL divergence in classical case via $\rho_C = \mathrm{diag}(\{P^x\})$

$\sigma_C = \mathrm{diag}(\{Q^x\})$

- Monotonicity $D(\rho||\sigma) \geq D\big(\mathcal{N}(\rho)||\mathcal{N}(\sigma)\big)$ under quantum channels $\mathcal{N}$

→ Unique such extension? If not, which one(s) to use?

From classical to quantum relative entropy

- KL divergence: $D(P||Q) = \sum_x P^x \log\left(\frac{P^x}{Q^x}\right)$
- Quantum relative entropy: $D(\rho||\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$
- Geometric relative entropy: $\widehat{D}(\rho||\sigma) = \mathrm{Tr}\left[\rho \log\left(\rho^{1/2}\sigma^{-1}\rho^{1/2}\right)\right]$
- Measured relative entropy via $\{M^x\}_x$ with $\mathbb{P}_{\rho,M}(x) = \mathrm{Tr}[M^x\rho]$ as

$$D_{\mathrm{ALL}}(\rho||\sigma) = \sup_M D\left(\mathbb{P}_{\rho,M}||\mathbb{P}_{\sigma,M}\right)$$

- Classical case, monotonicity, ordering: $\widehat{D}(\rho||\sigma) \geq D(\rho||\sigma) \geq D_{ALL}(\rho||\sigma)$

Further: locally measured relative entropies

- Bipartite system AB, local measurements with respect to $A{:}B$ as

$$M = \mathrm{LO}(A{:}B), \mathrm{LOCC}_1(\mathrm{A} \to B), \mathrm{LOCC}(A{:}B), \mathrm{SEP}(A{:}B), \mathrm{PPT}(A{:}B)$$

leads to

$$D_{\mathrm{M}}(\rho_{AB}||\sigma_{AB}) \coloneqq \sup_{M \in M} D(\mathbb{P}_{\rho,M}||\mathbb{P}_{\sigma,M})$$

with $\mathbb{P}_{\rho,M}(x) = \mathrm{Tr}[\rho M^x]$ from $\{M^x\}_x$ with $M^x \in M$

- Classical case, (restricted) monotonicity, ordering:

$$D_{\mathrm{ALL}}(\rho_{AB}||\sigma_{AB}) \geq D_{\mathrm{PPT}}(\rho_{AB}||\sigma_{AB}) \geq \cdots \geq D_{\mathrm{LO}}(\rho_{AB}||\sigma_{AB}) \geq 0$$

with $D_{\mathrm{M}}(\rho_{AB}||\sigma_{AB}) = 0 \Leftrightarrow \rho_{AB} = \sigma_{AB}$ (informationally complete)

Mathematical methods

Matrix analysis

- Multipartite quantum systems: entanglement leads to non-commutativity
- Matrix trace inequalities, e.g., for H_1, H_2 Hermitian:

$$\exp(H_1)\exp(H_2) = \exp(H_1 + H_2 + 0.5 * [H_1, H_2] + \cdots)$$ Baker-Campbell-Haus.

$$\mathrm{Tr}[\exp(H_1)\exp(H_2)] \geq \mathrm{Tr}[\exp(H_1 + H_2)]$$ Golden-Thompson

$$\exp(\mathrm{Tr}[\exp(H_1)H_2]) \leq \mathrm{Tr}[\exp(H_1 + H_2)] \text{ for } \mathrm{Tr}[\exp(H_1)] = 1$$ Peierls-Bogolyubov

- For example, entropy is non-negative:

$$H(A)_\rho = -\mathrm{Tr}[\rho_A \log(\rho_A)] \geq \log \mathrm{Tr}\left[\exp\left(\log(\rho_A) + \log(\rho_A)\right)\right] = -\log(\mathrm{Tr}[\rho_A^2]) \geq 0$$

- Challenge: multipartite systems ⇔ multivariate inequalities

Workhorse A: Multivariate trace inequalities

- Multipartite systems $ABC \Leftrightarrow$ non-commuting matrices $\rho_{ABC}, \rho_{AB}, \rho_{BC}, \cdots$
- Example trace inequality: Golden-Thompson for H_1, H_2 Hermitian as

$$\mathrm{Tr}[\exp(H_1 + H_2)] \leq \mathrm{Tr}[\exp(H_1)\exp(H_2)]$$

- Multivariate extension for $\{H_k\}_{k=1}^n$ Hermitian as

$$\log ||\exp(\sum_{k=1}^{n} H_k)||_p \leq \int_{-\infty}^{\infty} d\beta_0(t) \log ||\prod_{k=1}^{n} \exp\big((1+it)H_k\big)||_p$$

with probability distribution $\beta_0(t) := \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}$

→ Sutter, B., Tomamichel [CMP 2017]

Workhorse A: Example

- For $n = 2$ and $p = 2$ relaxed to Golden-Tompson inequality:

$$\mathrm{Tr}[\exp(H_1 + H_2)] \leq \mathrm{Tr}[\exp(H_1)\exp(H_2)]$$

- For $n = 3$ and $p = 2$ relaxed to Lieb's triple matrix inequality:

$$\mathrm{Tr}[\exp(H_1 + H_2 + H_3)] \leq \int_{-\infty}^{\infty} d\beta_0(t)\mathrm{Tr}[\exp(H_1)\exp\left(\tfrac{1+it}{2} H_2\right)\exp(H_3)\exp\left(\tfrac{1-it}{2} H_2\right)]$$

$$= \int_0^{\infty} \mathrm{d}\lambda\, \mathrm{Tr}\left[\exp(H_1)\tfrac{1}{\exp(-H_2+\lambda)}\exp(H_3)\,\tfrac{1}{\exp(-H_2+\lambda)}\right]$$

- Arbitrarily many matrices possible → treat multipartite quantum states!

Workhorse B: Variational formulas

- KL divergence $D(P||Q) = \sup_{W \geq 0} \sum_x P_x \log W_x - \log(\sum_x Q_x W_x)$
- Quantum relative entropy

$$D(\rho||\sigma) = \sup_{\omega \geq 0} \operatorname{Tr}[\rho \log \omega] - \log \operatorname{Tr}[\exp(\log \sigma + \log \omega)]$$

- Measured relative entropy

$$D_{ALL}(\rho||\sigma) = \sup_{\omega \geq 0} \operatorname{Tr}[\rho \log \omega] - \log \operatorname{Tr}[\sigma \omega]$$

- Theorem: locally measured relative entropy

$$D_M(\rho||\sigma) \leq \sup_{\omega \in C_M} \operatorname{Tr}[\rho \log \omega] - \log \operatorname{Tr}[\sigma \omega]$$

with $C_M := \bigcup_{M \in \mathbf{M}} \operatorname{cone}\{M_x\}$, equal. for $\mathbf{M} = \mathrm{LO}, \mathrm{LOCC}_1$ $(A \to B)$ via emb.

Example proof

Proof of quantum SSA

- Show that $I(A\colon B|C)_\rho = H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho \geq 0$
- Non-commutative case:

$$I(A\colon B|C)_\rho = D(\rho_{ABC}||\exp(\log\rho_{AC} + \log\rho_{BC} - \log\rho_{ABC}))$$

$$= \sup_{\omega_{ABC}\geq 0} \mathrm{Tr}[\rho_{ABC}\log\omega_{ABC}] - \log\mathrm{Tr}[\exp(\log\omega_{ABC} + \log\rho_{AC} + \log\rho_{BC} - \log\rho_C)]$$

$$\geq \sup_{\omega_{ABC}\geq 0} \mathrm{Tr}[\rho_{ABC}\log\omega_{ABC}] - \log\mathrm{Tr}[\omega_{ABC}\sigma_{ABC}]$$

$$= D_{\mathrm{M}}(\rho_{ABC}||\sigma_{ABC})$$

for $\sigma_{ABC} = \int_{-\infty}^{\infty} \mathrm{d}\beta_0(t)\, \rho_{AC}^{\frac{1+it}{2}} \rho_C^{\frac{-1+it}{2}} \rho_{BC} \rho_C^{\frac{-1-it}{2}} \rho_{AC}^{\frac{1-it}{2}}$ with $\sigma_{ABC} \geq 0$ and $\mathrm{Tr}[\sigma_{ABC}] = 1$

→ non-negativity of measured relative entropy for states $D_{\mathrm{M}}(\rho_{ABC}||\sigma_{ABC}) \geq 0$

Quantum entropy cone

We have no clue for $n \geq 4$!

- No further linear inequality known except for SSA (all there is for $n = 3$)
- Conjecture for ρ_{ABCD}, does it (still) hold that

$$2I(C\colon D)_\rho \leq I(A\colon B)_\rho + I(A\colon CD)_\rho + 3I(C\colon D|A)_\rho + I(C\colon D|B)_\rho \, ?$$

- More generally, due to monogamy of entanglement, any balanced non-Shannon type inequality (still) true?

→ Linden & Winter [CMP 2005]: we have constrained quantum inequalities,

e.g., $I(A\colon C|B)_\rho = 0 = I(B\colon C|A)_\rho$ implies

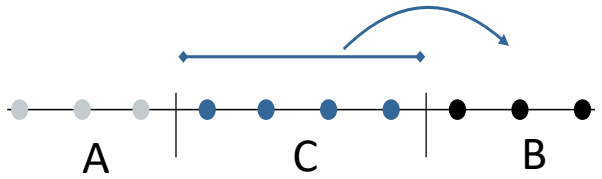
$$I(C\colon AB)_\rho \leq I(C\colon D)_\rho + I(A\colon B|D)_\rho$$

→ bottleneck are quantum SSA refinements $I(A\colon B|C)_\rho \geq f(\rho_{ABC})$

Quantum recoverability

Quantum SSA refinements: recoverability

- Non-commutative case:



$$I(A:B|C)_\rho = D(\rho_{ABC}||\exp(\log\rho_{AC} + \log\rho_{BC} - \log\rho_{ABC})) = ?$$

- Recoverability via quantum channels $R_{C\to BC}$ gives:

a. Equality condition: $I(A:B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = (I_A \otimes R_{C\to BC})(\rho_{AC})$

b. No-go: $I(A:B|C)_\rho \not\geq \inf_{R_{C\to BC}} D(\rho_{ABC}||(I_A \otimes R_{C\to BC})(\rho_{AC}))$

c. Theorem: $I(A:B|C)_\rho \geq \inf_{R_{C\to BC}} D_{ALL}(\rho_{ABC}||(I_A \otimes R_{C\to BC})(\rho_{AC}))$

→ first (weaker) version Fawzi & Renner [CMP 2015]
+ tight and direct matrix analysis proof Sutter, B., Tomamichel [CMP 2017]

Quantum Markov chains

Quantum SSA refinements: separability

- Quantum Markov chain (QMC) equality condition:

$$I(A\colon B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = \bigoplus_k p_k \, \rho^k_{AC^k_L} \otimes \rho^k_{C^k_R B} \text{ with } C = \bigoplus_k C^k_L \otimes C^k_R$$

- QMC no-go: $I(A\colon B|C)_\rho \not\geq \inf_{\sigma \in QMC(A-C-B)} D_{ALL}(\rho_{ABC}||\sigma_{ABC})$
- Separability no-go: $I(A\colon B|C)_\rho \not\geq \inf_{\sigma \in SEP(A:B)} D_{ALL}(\rho_{AB}||\sigma_{AB})$
- Separability theorem: $I(A\colon B|C)_\rho \geq \inf_{\sigma \in SEP(A:B)} D_{LOCC_1(B\to A)}(\rho_{AB}||\sigma_{AB})$

→ Brandão, Christandl, Yard [CMP 2011] (flawed)
+ tight and direct matrix analysis proof B., Tomamichel [CMP 2024]

Quantum SSA refinements: Markov chains

- Separability theorem:

$$I(A\colon B|C)_\rho \geq \inf_{\sigma\in \mathrm{SEP}(A:B)} D_{\mathrm{LOCC}_1(B\to A)}(\rho_{AB}||\sigma_{AB})$$

- QMC no-go:

$$I(A\colon B|C)_\rho \not\geq \inf_{\sigma\in \mathrm{QMC}(A-C-B)} D_{\mathrm{ALL}}(\rho_{ABC}||\sigma_{ABC})$$

- Conjectured QMC version:

$$I(A\colon B|C)_\rho \geq \inf_{\sigma\in \mathrm{QMC}(A-C-B)} D_{\mathrm{LO}(A:B:C)}(\rho_{ABC}||\sigma_{ABC})\,?$$

- We have no clue → bottleneck are equality conditions:

$$I(A\colon B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = \bigoplus_k p_k\ \rho^k_{AC^k_L} \otimes \rho^k_{C^k_R B} \text{ with } C = \bigoplus_k C^k_L \otimes C^k_R$$

Conclusions

- Extending classical entropy inequalities to quantum states gives entanglement monogamy inequalities

  → reveals the structure of multipartite quantum states

- Good to have different proofs of results (some proofs might be flawed!)

  → matrix analysis directly leads to tight and most general bounds

- Enablers:
  1) Locally measured quantum relative entropies
  2) Variational formulas for quantum entropies
  3) Multivariate matrix trace inequalities

Open questions include

- Quantum entropy versus quantum Markov chains (QMC)?
- Quantum entropy cone?
- Entropic uncertainty relations for sets of multiple measurements?
- Many more…
- Applications throughout science!

  - Postdoc and PhD positions at RWTH Aachen University
  - Get in contact:

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