Entanglement monogamy via multivariate trace inequalities

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Matrix analysis for quantum entropy inequalities

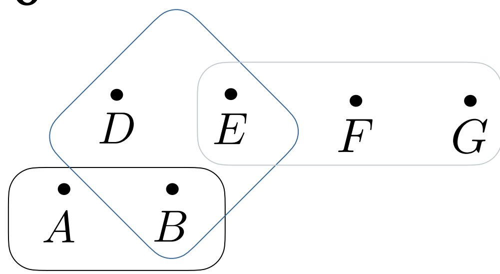
Entropy inequalities

- Probability distribution P_{ABC} with entropy $H(A)_P \coloneqq -\sum_x P_A^x \log P_A^x \geq 0$
- Mutual information:

$$I(A{:}B)_P \coloneqq H(A)_P + H(B)_P - H(AB)_P \geq 0$$

- Conditional entropy:

$$H(A|B)_P \coloneqq H(AB)_P - H(B)_P \geq 0$$



- Conditional mutual information:

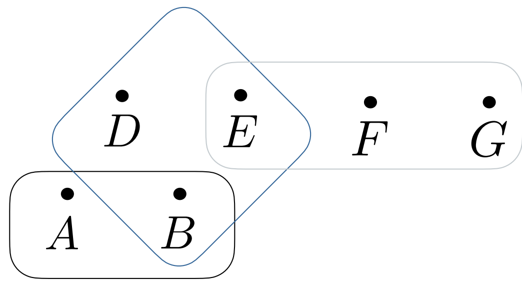
$$I(A{:}B|C)_P \coloneqq H(AC)_P + H(BC)_P - H(C)_P - H(ABC)_P \geq 0$$

- Non-Shannon type inequalities (!) – entropy cone, e.g.,

$$2I(C{:}D)_P \leq I(A{:}B)_P + I(A{:}CD)_P + 3I(C{:}D|A)_P + I(C{:}D|B)_P$$

Quantum entropy inequalities

- State $\rho_{ABC} \succcurlyeq 0$, $\mathrm{Tr}[\rho_{ABC}] = 1$ with entropy $H(A)_\rho \coloneqq -\mathrm{Tr}[\rho_A \log \rho_A] \geq 0$
- $I(A:B)_\rho := H(A)_\rho + H(B)_\rho - H(AB)_\rho \geq 0$
- $H(A|B)_\rho \coloneqq H(AB)_\rho - H(B)_\rho$ can get negative
- Strong subadditivity (SSA):



$$I(A:B|C)_\rho \coloneqq H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho \geq 0$$

- Equivalent to entanglement monogamy = weak monotonicity:

$$H(A|C)_\rho + H(A|B)_\rho \geq 0$$

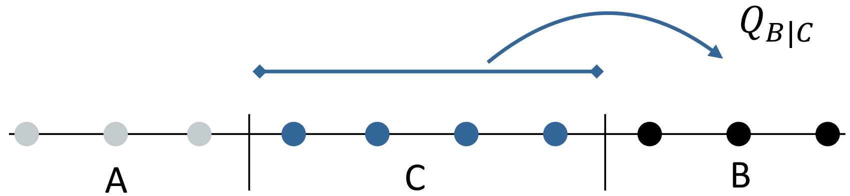
- Today: Prove (novel) quantum entropy inequalities via matrix analysis

Refined entropy inequalities beyond SSA

SSA refinements

- Recoverability via parent relative entropy $D(P||Q) := \sum_x P^x \log\left(\frac{P^x}{Q^x}\right)$:

$$I(A\colon B|C)_P = \inf_{Q_{B|C}} D\left(P_{ABC}||Q_{B|C}P_{AC}\right) \geq 0 \text{ with } Q_{BC} = P_{BC}P_c^{-1}$$



- Equivalent Markov chain formulation:

$$I(A\colon B|C)_P = \inf_{Q_{A-c-B}} D(P_{ABC}||Q_{A-C-B}) \geq 0 \text{ with } Q_{A-C-B} = P_{BC}P_C^{-1}P_{AC}$$

- Equality conditions $I(A\colon B|C)_P = 0$ from $D(P||Q) = 0 \Leftrightarrow P = Q$

Quantum versions?

From classical to quantum relative entropy

- Relative entropy: $D(P||Q) = \sum_x P^x \log\left(\frac{P^x}{Q^x}\right)$
- Quantum relative entropy: $D(\rho||\sigma) \coloneqq \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$
- Geometric relative entropy: $\widehat{D}(\rho||\sigma) \coloneqq \mathrm{Tr}\left[\rho\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})\right]$
- Measured relative entropy for POVMs $\{M^x\}$ with $\mathbb{P}_{\rho,M}(x) \coloneqq \mathrm{Tr}[M^x\rho]$ as

$$D_{ALL}(\rho||\sigma) \coloneqq \sup_M D\left(\mathbb{P}_{\rho,M}||\mathbb{P}_{\sigma,M}\right)$$

- Good properties, ordering: $\widehat{D}(\rho||\sigma) \geq D(\rho||\sigma) \geq D_{ALL}(\rho||\sigma)$

Variant: Locally measured relative entropies

- Bipartite system AB, local measurements with respect to $A:B$ as

$$M = LO(A:B), \mathrm{LOCC}_1(\mathrm{A} \rightarrow B), LOCC(A:B), SEP(A:B), PPT(A:B)$$

leads to

$$D_M(\rho_{AB}||\sigma_{AB}) := \sup_{M \in M} D(\mathbb{P}_{\rho,M}||\mathbb{P}_{\sigma,M})$$

with $\mathbb{P}_{\rho,M}(x) = \mathrm{Tr}[\rho M^x]$ from POVMs $\{M^x\}$ with $M^x \in M$

- Reasonable properties, ordering:

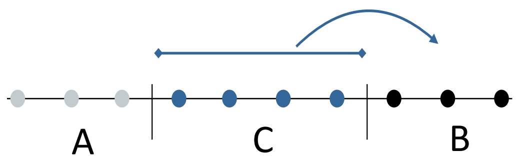
$$D_{ALL}(\rho_{AB}||\sigma_{AB}) \geq D_{PPT}(\rho_{AB}||\sigma_{AB}) \geq \cdots \geq D_{LO}(\rho_{AB}||\sigma_{AB}) \geq 0$$

with $D_M(\rho_{AB}||\sigma_{AB}) = 0 \Leftrightarrow \rho_{AB} = \sigma_{AB}$ (informationally complete)

Quantum recoverability

Quantum SSA refinements: Recoverability

- Non-commutative rewriting:



$$I(A:B|C)_\rho = D(\rho_{ABC}||\exp(\log\rho_{AC} + \log\rho_{BC} - \log\rho_{ABC})) = ???$$

- Recoverability via quantum channels $R_{C\to BC}$ gives:

a. Equality condition: $I(A:B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = (I_A \otimes R_{C\to BC})(\rho_{AC})$

b. No-go: $I(A:B|C)_\rho \ngeq \inf_{R_{C\to BC}} D(\rho_{ABC}||(I_A \otimes R_{C\to BC})(\rho_{AC}))$

c. Theorem: $I(A:B|C)_\rho \geq \inf_{R_{C\to BC}} D_{ALL}(\rho_{ABC}||(I_A \otimes R_{C\to BC})(\rho_{AC}))$

→ QIP 2016 [Fawzi, Renner] + many more, direct matrix analysis proof

QIP 2017 [Sutter, B., Tomamichel]

Quantum Markov chains

Quantum SSA refinements: Separability

- Exact quantum Markov chain (QMC) equality condition:

  $I(A:B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = \bigoplus_k p_k\ \rho^k_{AC^k_L} \otimes \rho^k_{C^k_R B}$ with $C = \bigoplus_k C^k_L \otimes C^k_R$

- QMC no-go: $I(A:B|C)_\rho \ngeq \inf_{\sigma \in QMC(A-C-B)} D_{ALL}(\rho_{ABC}||\sigma_{ABC})$

- Separability no-go: $I(A:B|C)_\rho \ngeq \inf_{\sigma \in SEP(A:B)} D_{ALL}(\rho_{AB}||\sigma_{AB})$

- Theorem: $I(A:B|C)_\rho \geq \inf_{\sigma \in SEP(A:B)} D_{LOCC_1(B\to A)}(\rho_{AB}||\sigma_{AB})$

  QIP 2011 [Brandão, Christandl, Yard]

  → Today: Give novel matrix analysis proof of strengthened bounds

Proof techniques

Workhorse A: Multivariate trace inequalities

- Multipartite systems $ABC \Leftrightarrow$ non-commuting matrices $\rho_{ABC}, \rho_{AB}, \rho_{BC}, \cdots$
- Example trace inequality: Golden-Thompson for H_1, H_2 Hermitian as

$$\mathrm{Tr}[\exp(H_1 + H_2)] \leq \mathrm{Tr}[\exp(H_1)\exp(H_2)]$$

- Multivariate extension for $\{H_k\}_{k=1}^{n}$ Hermitian as

$$\log || \exp(\sum_{k=1}^{n} H_k)||_p \leq \int_{-\infty}^{\infty} d\beta_0(t) \log || \prod_{k=1}^{n} \exp\big((1+it)H_k\big) ||_p$$

with probability distribution $\beta_0(t) := \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}$

→ QIP 2017 [Sutter, B., Tomamichel] for recoverability SSA strengthening

Workhorse B: Variational formulas

- Relative entropy $D(P||Q) = \sup_{W \geq 0} \sum_x P_x \log W_x - \log(\sum_x Q_x W_x)$
- Quantum relative entropy

$$D(\rho||\sigma) = \sup_{\omega \geq 0} \operatorname{Tr}[\rho \log \omega] - \log \operatorname{Tr}[\exp(\log \sigma + \log \omega)]$$

- Measured relative entropy (Golden-Thompson!)

$$D_{ALL}(\rho||\sigma) = \sup_{\omega \geq 0} \operatorname{Tr}[\rho \log \omega] - \log \operatorname{Tr}[\sigma \omega]$$

- Theorem: Locally measured relative entropy

$$D_{M}(\rho||\sigma) \leq \sup_{\omega \in C_M} \operatorname{Tr}[\rho \log \omega] - \log \operatorname{Tr}[\sigma \omega]$$

with $C_M := \bigcup_{M \in M} cone\{M_x\}$ and equality for $M = LO, LOCC_1(A \to B)$

Result: SSA separability refinements

- Theorem: $I(A:B|C)_\rho \geq E_{LOCC_1(B\to A)}(A:B) - \{E(A:C)_\rho - E_{ALL}(A:C)\}$

with shorthand notation relative entropies of entanglement

$$E(A:B) \coloneqq \inf_{\sigma\in SEP(A:B)} D(\rho_{AB}||\sigma_{AB}),\ E_M(A:B) \coloneqq \inf_{\sigma\in SEP(A:B)} D_M(\rho_{AB}||\sigma_{AB})$$

- Corollary: $I(A:B|C)_\rho \geq E^\infty_{LOCC_1(B\to A)}(A:B) \geq E_{LOCC_1(B\to A)}(A:B)$

via version on $\rho^{\otimes n}$ and asymptotic achievability

$$D^\infty_{ALL}(\rho||\sigma) \coloneqq \lim_{n\to\infty}\frac{1}{n}D_{ALL}\left(\rho^{\otimes n}||\sigma^{\otimes n}\right) = D(\rho||\sigma)$$

Proof sketch

Use variational formula

- Theorem: $I(A:B|C)_\rho \geq E_{LOCC_1(B\to A)}(A:B) - \{E(A:C)_\rho - E_{ALL}(A:C)\}$
- Proof via $I(A:B|C)_\rho + E(A:C)_\rho = D(\rho_{ABC}||\exp(\log\rho_{BC} + \log\sigma_{A:C} - \log\rho_C))$

  for $\sigma_{A:C} \coloneqq \mathrm{argmin}_{\sigma\in SEP(A:C)} D(\rho_{AC}||\sigma_{AC}) \equiv \sum_j \sigma_A^j \otimes \sigma_C^j$
- Variational formula for relative entropy on right-hand side gives

$$\sup_{\omega\geq 0} \mathrm{Tr}[\rho_{ABC}\log\omega_{ABC}] - \log\mathrm{Tr}[\exp(\log\omega_{ABC} + \log\rho_{BC} + \log\sigma_{A:C} - \log\rho_C))]$$

  and with choice $\omega_{ABC} = \exp(\log X_{B\to A} + \log Y_{AC}) \geq 0$ for $X_{B\to A} \in LOCC_1(B\to A)$

$$\sup_{X\in LOCC_1(B\to A), Y\geq 0} \mathrm{Tr}[\rho_{AB}\log X_{B\to A}] + \mathrm{Tr}[\rho_{AC}\log Y_{AC}] - \log\mathrm{Tr}[\exp(5\text{ summands})]$$

Use multivariate trace inequality

- 5-matrix Golden-Thompson inequality for $p = 2$ leads to:

$$\sup_{X \in LOCC_1(B\to A), Y \geq 0} \mathrm{Tr}[\rho_{AB} \log X_{B\to A}] + \mathrm{Tr}[\rho_{AC} \log Y_{AC}] - \log \mathrm{Tr}\left[Y_{AC} \int_{-\infty}^{\infty} d\beta_0(t)\, X_{B\to A}^{\frac{1+it}{2}} \gamma_{A:BC}^{t} X_{B\to A}^{\frac{1-it}{2}}\right]$$

with $\gamma_{A:BC}^{t} = \sum_j \sigma_A^j \otimes \left(\rho_{BC}^{\frac{1+it}{2}} \rho_C^{\frac{-1-it}{2}} \sigma_C^j \rho_C^{\frac{-1+it}{2}} \rho_{BC}^{\frac{1-it}{2}}\right) \in SEP(A{:}\,BC)$

- With $\gamma_{A:BC} := \int_{-\infty}^{\infty} d\beta_0\,(t)\, \gamma_{A:BC}^{t}$ rewritten as

$$\sup_{X \in LOCC_1(B\to A), Y \geq 0} \mathrm{Tr}[\rho_{AB} \log X_{B\to A}] + \mathrm{Tr}[\rho_{AC} \log Y_{AC}] - \log \mathrm{Tr}[X_{B\to A}\gamma_{A:B}] \cdot \mathrm{Tr}[Y_{AC}\hat{\gamma}_{A:C}]$$

and if $\hat{\gamma}_{A:C} := \dfrac{\mathrm{Tr}_B\left[\int_{-\infty}^{\infty} d\beta_0\,(t) X_{B\to A}^{\frac{1+it}{2}} \gamma_{A:BC}^{t} X_{B\to A}^{\frac{1-it}{2}}\right]}{Tr[X_{B\to A}\gamma_{A:B}]} \in SEP(A{:}\,C)$ this would give the sought-after lower bound $E_{LOCC_1(B\to A)}(A{:}\,B)_\rho + E_{ALL}(A{:}\,C)_\rho$

Use one-way LOCC property

- Altogether: $I(A:B|C)_\rho + E(A:C)_\rho \geq E_{LOCC_1(B\to A)}(A:B)_\rho + E_{ALL}(A:C)_\rho$
- Comments:
  - For the last step $\hat{\gamma}_{A:C} := \frac{\mathrm{Tr}_B\left[\int_{-\infty}^{\infty} d\beta_0\,(t) X_{B\to A}^{\frac{1+it}{2}} \gamma_{A:BC}^{t} X_{B\to A}^{\frac{1-it}{2}}\right]}{Tr[X_{B\to A}\gamma_{A:B}]} \in SEP(A:C)$ it is crucial that $X_{B\to A} \in LOCC_1(B \to A)$ and thus $E_{LOCC_1(B\to A)}(A:B)_\rho$ appears in lower bound
  - Follows as wlog $X_{B\to A} = \sum_x \omega_A^x \otimes P_B^x$ with $\{P_B^x\}$ set of mutually orthogonal projectors (for B large enough), phases $(1 \pm it)/2$ from multivariate cancel!
  - Inequalities used are:
    1. 5-matrix Golden-Thompson
    2. Interaction Hamiltonian $\omega_{ABC} = \exp(\log X_{B\to A} + \log Y_{AC})$ with $X_{B\to A} \in LOCC_1(B \to A)$

Extensions

Further results

- Generalized entanglement monogamy inequalities, e.g.,

$$E(A:BC)_\rho \geq E_{LOCC_1(B\to A)}(A:B) + E_{ALL}(A:C)_\rho$$

- Conditional entanglement of mutual information (CEMI) for $\rho_{A\bar{A}B\bar{B}}$ gives:
  - Strengthened separability refinement $E_{SEP}(A:B)_\rho$
  - PPT separability refinements $P_{PPT}(A:B)_\rho$
- Multipartite separability refinements via n-matrix Golden-Thompson
- Multipartite squashed entanglement and multipartite CEMI lead to corresponding recoverability refinements

Conclusions

- Extending classical entropy inequalities to quantum states gives entanglement monogamy inequalities

  → reveals the structure of multipartite quantum states

- Good to have different proofs of results (some proofs might be flawed!)

  → matrix analysis directly leads to tight and all most general bounds

- Enablers:
  1) Locally measured quantum relative entropies
  2) Variational formulas for quantum entropies
  3) Multivariate matrix trace inequalities

Where to go from here

- Entropic uncertainty relations for sets of multiple measurements?
- Faithfulness of multipartite squashed entanglement?
- Quantum entropy versus exact quantum Markov chains (QMC)?
- Quantum entropy cone?
- Upcoming at QIP 2023 (list incomplete):
  - *56 Quantum Rényi and f-divergences from integral representations*
  - *67 Connecting entanglement distillation and entanglement testing with restricted measurements*
  - *85 The Quantum Entropy Cone near its Apex*
  - *96 Clustering of conditional mutual information and quantum Markov structure at arbitrary temperatures*
  - *449 A new operator extension of strong subadditivity of quantum entropy*

I am hiring

- Postdoc and PhD positions at RWTH Aachen University
- Get in contact:

  berta@physik.rwth-aachen.de

- **Thank you, questions?**

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