Broadcast Channel Simulation

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***Abstract*—We study the problem of random-assisted simulation of discrete broadcast channel in one-shot and i.i.d. setups. We derive one-shot inner and outer bounds of the set of attainable message-size pairs for simulating $W_{\mathsf{YZ|X}}$ within some total variation distance (TVD) tolerance of ϵ. The inner bounds are based on the bipartite convex split lemma. Whereas the outer bounds are based on the properties of the multi-partite max information. Using these bounds, we establish a single-letter expression of the *simulation region* of a broadcast channel.**

I. Introduction

Channel simulation is a fundamental task in information theory, and is the reverse of the task of channel coding (*e.g.*, see [1], [2], [3]). It has been studied intensively in asymptotic setups in the context of the reverse Shannon theorem [4], [5], as well as in one-shot setups [6], [7]. In this paper, we consider the problem of simulating broadcast channels with unconstrained shared randomness between the sender and each of the receivers.

Given a two-receiver broadcast channel $W_{\mathsf{YZ|X}} \in \mathcal{P}(\mathcal{Y} \times \mathcal{Z}|\mathcal{X})$ and a pair of shared random variables S and S' on set $\mathcal{S}$ and $\mathcal{S}'$, respectively, the task of simulating $W_{\mathsf{YZ|X}}$ within a tolerance of ϵ in the total variation distance (TVD) is to find a pair of sets of encoders $\mathcal{E}_s^Y \in \mathcal{P}(1,\dots,M|\mathcal{X})$, $\mathcal{E}_{s'}^Z \in \mathcal{P}(1,\dots,N|\mathcal{X})$ and a pair of sets of decoders $\mathcal{D}_s^Y \in \mathcal{P}(\mathcal{Y}|1,\dots,M)$, $\mathcal{D}_{s'}^Z \in \mathcal{P}(\mathcal{Z}|1,\dots,N)$ for each $s \in \mathcal{S}$ and $s' \in \mathcal{S}'$ such that, for all input distributions $p_{\mathsf{X}} \in \mathcal{P}(\mathcal{X})$, the induced joint distribution

$$\begin{aligned}\tilde{p}_{\mathsf{XYZ}}(x,y,z) \coloneqq \sum_{s\in\mathcal{S},s'\in\mathcal{S}'} p_{\mathsf{S}}(s)\cdot p_{\mathsf{S}'}(s')\cdot p_{\mathsf{X}}(x)\cdot \\ \sum_{\substack{m\in\{1,\dots,M\}\\ n\in\{1,\dots,N\}}} \mathcal{E}_s^Y(m|x)\cdot\mathcal{E}_{s'}^Z(n|x)\cdot\mathcal{D}_s^Y(y|m)\cdot\mathcal{D}_{s'}^Z(z|n)\end{aligned} \tag{1}$$

is ϵ-close to the original joint distribution $p_{\mathsf{XYZ}}(x,y,z) \coloneqq p_{\mathsf{X}}(x) \cdot W_{\mathsf{YZ|X}}(y,z|x)$ in TVD. We call the 5-tuple $(\{\mathcal{E}_s^Y\}_{s\in\mathcal{S}}, \{\mathcal{E}_{s'}^Z\}_{s'\in\mathcal{S}'}, \{\mathcal{S}_s^Y\}_{s\in\mathcal{S}}, \{\mathcal{S}_{s'}^Z\}_{s'\in\mathcal{S}'}, \mathsf{S})$ a size-(M,N) ϵ-simulation code for $W_{\mathsf{YZ|X}}$ (see Fig. 1).

A message-size pair (M,N) is said to be ϵ-attainable if there exists a size-(M,N) ϵ-simulation code, and we denote $\mathcal{M}_\epsilon^\star(W_{\mathsf{YZ|X}})$ the set of all ϵ-attainable message-size pairs. In Section II, we propose a pair of subset and superset for $\mathcal{M}_\epsilon^\star(W_{\mathsf{YZ|X}})$. The inner (achievability) bound is based on the bipartite convex split lemma [8], and the outer (converse) bound is based on various properties of multipartite max-information.

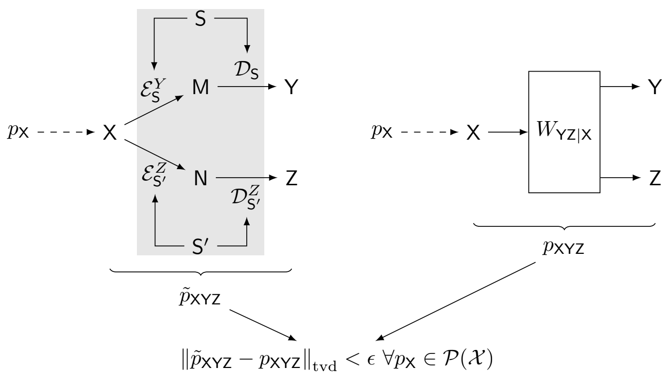

Fig. 1. The task of simulating a bipartite broadcast channel with the help of common randomness between the sender and the two receivers, respectively, under the worst case error criteria given by the channel TVD.

We are also interested in the problem of simulating n identical copies of a broadcast channel, *i.e.*, $W_{\mathsf{YZ|X}}^{\otimes n}$. In Section III, we study the first-order asymptotic limit of the points in $\mathcal{M}_\epsilon^\star(W_{\mathsf{YZ|X}}^{\otimes n})$, *i.e.*, the set containing the limits of $(\frac{1}{n}\log M, \frac{1}{n}\log N)$ for ϵ-attainable message-size pairs (M,N) for $W_{\mathsf{YZ|X}}^{\otimes n}$ (known as the *simulation region* as defined in (21)). In contrast to the channel coding problem of broadcast channels, using the one-shot results in Section II, we manage to derive a single-letter expression of the simulation region.

A more comprehensive version of this paper is available at [9].

Notations

- Bold symbols are vectors, *e.g.*, $\boldsymbol{x}_1^n \equiv (x_1,\dots,x_n)$.
- Given two channels $W_{\mathsf{Y|X}}$ and $\tilde{W}_{\mathsf{Y|X}}$, the TVD between them is defined as
$$\left\|\tilde{W}_{\mathsf{Y|X}} - W_{\mathsf{Y|X}}\right\|_{\mathrm{tvd}} \coloneqq \sup_{x\in\mathcal{X}} \left\|\tilde{W}_{\mathsf{Y|X}}(\cdot|x) - W_{\mathsf{Y|X}}(\cdot|x)\right\|_{\mathrm{tvd}}. \tag{2}$$
- The (truncated) spectral divergence of distributions p *v.s.* q on a same alphabet $\mathcal{X}$ is defined as
$$D_{s+}^\epsilon(p\|q) \coloneqq \inf\left\{a \geqslant 0 : \Pr_{\mathsf{X}\sim p}\left[\log\frac{p(\mathsf{X})}{q(\mathsf{X})} > a\right] < \epsilon\right\} \tag{3}$$
where $\epsilon \in (0,1)$.
- The *max-divergence* of distributions p w.r.t. q on a same alphabet $\mathcal{X}$ is defined as
$$D_{\max}(p\|q) \coloneqq \sup_{x\in\mathcal{X}}\left\{\log\frac{p(x)}{q(x)}\right\}. \tag{4}$$

- Let XY be random variables jointly distributed according to p_{XY}, the max information between X and Y is defined as
$$I_{\max}(\mathsf{X}:\mathsf{Y}) := \inf_{q_Y} D_{\max}(p_{\mathsf{XY}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}). \quad (5)$$
- Let XYZ be random variables jointly distributed according to p_{XYZ}, the common information among X, Y, and Z is defined as
$$I(\mathsf{X}:\mathsf{Y}:\mathsf{Z}) := H(\mathsf{X}) + H(\mathsf{Y}) + H(\mathsf{Z}) - H(\mathsf{XYZ}). \quad (6)$$
The max information among X, Y, and Z is defined as
$$I_{\max}(\mathsf{X}:\mathsf{Y}:\mathsf{Z}) = \inf_{q_{\mathsf{Y}}} \inf_{r_{\mathsf{Z}}} D_{\max}(p_{\mathsf{XYZ}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}} \times r_{\mathsf{Z}}). \quad (7)$$
- For bipartite broadcast channel $W_{\mathsf{YZ|X}}$, we define
$$\tilde{C}(W_{\mathsf{YZ|X}}) := \sup_{p_{\mathsf{X}} \in \mathcal{P}(\mathcal{X})} I(\mathsf{X}:\mathsf{Y}:\mathsf{Z})_{p_{\mathsf{X}} \cdot W_{\mathsf{YZ|X}}}. \quad (8)$$

II. One-Shot Simulation Region of Broadcast Channels

A. One-Shot Achievability Bound

Lemma 1 (Bipartite Convex Split Lemma [8, Fact 7, slightly modified])**.** *Let $\epsilon, \delta_1, \delta_2, \delta_3 \in (0,1)$, such that $\delta_1^2 + \delta_2^2 + \delta_3^2 \leqslant \epsilon^2$. Let $(\mathsf{X}, \mathsf{Y}, \mathsf{Z})$ be jointly distributed over $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ with pmf p_{XYZ}. Let q_{Y} and r_{Z} be two pmfs over $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Let M and N be two positive integers such that*
$$\log M \geqslant D_{s+}^{\epsilon_1}(p_{\mathsf{XY}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}) - \log \delta_1^2, \quad (9)$$
$$\log N \geqslant D_{s+}^{\epsilon_2}(p_{\mathsf{XZ}} \| p_{\mathsf{X}} \times r_{\mathsf{Z}}) - \log \delta_2^2, \quad (10)$$
$$\log M + \log N \geqslant D_{s+}^{\epsilon_3}(p_{\mathsf{XYZ}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}} \times r_{\mathsf{Z}}) - \log \delta_3^2, \quad (11)$$
for some $\epsilon_1, \epsilon_2, \epsilon_3 \in (0,1)$ such that $\epsilon_1 + \epsilon_2 + \epsilon_3 \leqslant \epsilon - \sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2}$. Let J and K be independently uniformly distributed on $\{1, \ldots, M\}$ and $\{1, \ldots, N\}$, respectively. Let joint random variables $(\mathsf{J}, \mathsf{K}, \mathsf{X}, \mathsf{Y}_1, \ldots, \mathsf{Y}_M, \mathsf{Z}_1, \ldots, \mathsf{Z}_N)$ be distributed according to
$$p_{\mathsf{X},\mathsf{Y}_1,\ldots,\mathsf{Y}_M,\mathsf{Z}_1,\ldots,\mathsf{Z}_N|\mathsf{J},\mathsf{K}}(x, y_1, \ldots, y_M, z_1, \ldots, z_N | j,k) = p_{\mathsf{XYZ}}(x, y_j, z_k) \cdot \prod_{i \neq j} q_{\mathsf{Y}}(y_i) \cdot \prod_{\ell \neq k} r_{\mathsf{Z}}(z_\ell). \quad (12)$$
Then,
$$\| p_{\mathsf{X},\mathsf{Y}_1,\ldots,\mathsf{Y}_M,\mathsf{Z}_1,\ldots,\mathsf{Z}_N} - p_{\mathsf{X}} \times q_{\mathsf{Y}_1} \times \cdots \times q_{\mathsf{Y}_M} \times r_{\mathsf{Z}_1} \times \cdots \times r_{\mathsf{Z}_N} \|_{\mathrm{tvd}} \leqslant \epsilon. \quad (13)$$

Theorem 2. *Let $W_{\mathsf{YZ|X}}$ be a DMC from $\mathcal{X}$ to $\mathcal{Y} \times \mathcal{Z}$, and let $\epsilon \in (0,1)$. For any $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that $\epsilon_1 + \epsilon_2 + \epsilon_3 \leqslant \epsilon$, $\delta_1 \in (0, \epsilon_1)$, $\delta_2 \in (0, \epsilon_2)$, $\delta_3 \in (0, \epsilon_3)$, $q_{\mathsf{Y}} \in \mathcal{P}(\mathcal{Y})$, and $r_{\mathsf{Z}} \in \mathcal{P}(\mathcal{Z})$, the following set is a subset of $\mathcal{M}_\epsilon^\star(W_{\mathsf{YZ|X}})$*
$$\mathcal{M}_\epsilon^{in}(W_{\mathsf{YZ|X}}) := \left\{ (M,N) \in \mathbb{Z}_{>0}^2 : \begin{array}{r} \log M > D_{s+}^{\epsilon_1 - \delta_1}(p_{\mathsf{X}} \cdot W_{\mathsf{Y|X}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}) - \log \delta_1^2 \\ \log N > D_{s+}^{\epsilon_2 - \delta_2}(p_{\mathsf{X}} \cdot W_{\mathsf{Z|X}} \| p_{\mathsf{X}} \times r_{\mathsf{Z}}) - \log \delta_2^2 \\ \log MN > D_{s+}^{\epsilon_3 - \delta_3}(p_{\mathsf{X}} \cdot W_{\mathsf{YZ|X}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}} \times r_{\mathsf{Z}}) - \log \delta_3^2 \\ \forall p_{\mathsf{X}} \in \mathcal{P}(\mathcal{X}) \end{array} \right\} \quad (14)$$
where the reduced channels $W_{\mathsf{Y|X}}$ and $W_{\mathsf{Z|X}}$ are defined as
$$W_{\mathsf{Y|X}}(y|x) := \sum_{z \in \mathcal{Z}} W_{\mathsf{YZ|X}}(y, z|x), \quad (15)$$
$$W_{\mathsf{Z|X}}(y|x) := \sum_{y \in \mathcal{Y}} W_{\mathsf{YZ|X}}(y, z|x). \quad (16)$$

Proof. For arbitrary $q_{\mathsf{Y}} \in \mathcal{P}(\mathcal{Y})$, $r_{\mathsf{Z}} \in \mathcal{P}(\mathcal{Z})$, we present a protocol for simulating $W_{\mathsf{YZ|X}}$ by sending messages with alphabet sizes M and N to each of the receivers, respectively, where (M,N) is any integer pair satisfying the conditions on the RHS of (14). The protocol is as follows:

1) Let the sender and first receiver share i.i.d. random variables $(\mathsf{Y}_1, \ldots, \mathsf{Y}_M)$ where $\mathsf{Y}_k \sim q_{\mathsf{Y}}$ for each k.
2) Let the sender and second receiver share i.i.d. random variables $(\mathsf{Z}_1, \ldots, \mathsf{Z}_N)$ where $\mathsf{Z}_k \sim r_{\mathsf{Z}}$ for each k.
3) Upon receiving input $\mathsf{X} = x$, the sender generates a pair of random variables J, K (distributed on $\{1, \ldots, M\} \times \{1, \ldots, N\}$) according to the following conditional pmf
$$\tilde{p}_{\mathsf{J},\mathsf{K}|\mathsf{X},\mathbf{Y}_1^M,\mathbf{Z}_1^N}(j,k|x,\boldsymbol{y}_1^M,\boldsymbol{z}_1^N) \propto W_{\mathsf{YZ|X}}(y_j, z_k|x) \cdot \prod_{i \neq j} q_{\mathsf{Y}}(y_i) \cdot \prod_{\ell \neq k} r_{\mathsf{Z}}(z_\ell).$$
4) The sender sends J and K losslessly to the first and the second receiver using $\log M$-bits and $\log N$-bits, respectively.
5) Upon receiving J, the first sender outputs Y_{J}.
6) Upon receiving K, the second sender outputs Z_{K}.

It suffices to show the joint pmf of the random variables $\mathsf{XY_JZ_K}$ generated by the above protocol is ϵ-close (in TVD) to $p_{\mathsf{XYZ}} := p_{\mathsf{X}} \cdot W_{\mathsf{YZ|X}}$ for any input distribution p_{X}.

Let $\tilde{p}$ denote (joint/marginal/conditional, depending on the subscript) pmfs of the random variables J, K, X, $\mathsf{Y}_1, \ldots, \mathsf{Y}_M$, $\mathsf{Z}_1, \ldots, \mathsf{Z}_N$ as in the above protocol. Define the joint pmf $p_{\mathsf{J},\mathsf{K},\mathsf{X},\mathsf{Y}_1,\ldots,\mathsf{Y}_M,\mathsf{Z}_1,\ldots,\mathsf{Z}_N}$ as
$$p_{\mathsf{J},\mathsf{K},\mathsf{X},\mathbf{Y}_1^M,\mathbf{Z}_1^N}(j,k,x,\boldsymbol{y}_1^M,\boldsymbol{z}_1^N) := \frac{p_{\mathsf{X},\mathbf{Y}_1^M,\mathbf{Z}_1^N|\mathsf{J},\mathsf{K}}(x,\boldsymbol{y}_1^M,\boldsymbol{z}_1^N|j,k)}{M \cdot N},$$
where $p_{\mathsf{X},\mathbf{Y}_1^M,\mathbf{Z}_1^N|\mathsf{J},\mathsf{K}}$ has been defined in (12). For any input distribution, by definition, it holds that $p_{\mathsf{X}} = \tilde{p}_{\mathsf{X}}$. As a direct result of the protocol, we have
$$\tilde{p}_{\mathsf{J},\mathsf{K},\mathsf{X},\mathbf{Y}_1^M,\mathbf{Z}_1^N}(j,k,x,\boldsymbol{y}_1^M,\boldsymbol{z}_1^N) = p_{\mathsf{X}}(x) \cdot \prod_{i=1}^M q_{\mathsf{Y}}(y_i) \cdot \prod_{\ell=1}^N r_{\mathsf{Z}}(z_\ell) \cdot \frac{W_{\mathsf{YZ|X}}(y_j, z_k|x) \cdot \prod_{i \neq j} q_{\mathsf{Y}}(y_i) \cdot \prod_{\ell \neq k} r_{\mathsf{Z}}(z_\ell)}{\sum_{j',k'} W_{\mathsf{YZ|X}}(y_{j'}, z_{k'}|x) \cdot \prod_{i \neq j'} q_{\mathsf{Y}}(y_i) \cdot \prod_{\ell \neq k'} r_{\mathsf{Z}}(z_\ell)}.$$
By Lemma 1 and the requirements we imposed on M and N at the beginning of this proof (note that $\epsilon_1 - \delta_1 + \epsilon_2 - \delta_2 + \epsilon_2 - \delta_2 \leqslant \epsilon - \sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2}$), it holds that
$$\left\| \tilde{p}_{\mathsf{X}\mathbf{Y}_1^M\mathbf{Z}_1^N} - p_{\mathsf{X}\mathbf{Y}_1^M\mathbf{Z}_1^N} \right\|_{\mathrm{tvd}} = \left\| p_{\mathsf{X}} \cdot \prod_{i=1}^M q_{\mathsf{Y}_i} \cdot \prod_{\ell=1}^N r_{\mathsf{Z}_\ell} - \frac{1}{M \cdot N} \cdot p_{\mathsf{X}} \cdot \sum_{j,k} W_{\mathsf{Y}_j\mathsf{Z}_k|\mathsf{X}} \cdot \prod_{i \neq j} q_{\mathsf{Y}_i} \prod_{\ell \neq k} r_{\mathsf{Z}_\ell} \right\|_{\mathrm{tvd}} \leqslant \epsilon.$$

Since $\tilde{p}_{\mathsf{J},\mathsf{K}|\mathsf{X},\mathbf{Y}_1^M,\mathbf{Z}_1^N} = p_{\mathsf{J},\mathsf{K}|\mathsf{X},\mathbf{Y}_1^M,\mathbf{Z}_1^N}$ (as deliberately designed), we have

$$\left\|\tilde{p}_{\mathsf{JKX}\mathbf{Y}_1^M\mathbf{Z}_1^N} - p_{\mathsf{JKX}\mathbf{Y}_1^M\mathbf{Z}_1^N}\right\|_{\mathrm{tvd}} = \left\|\tilde{p}_{\mathsf{X}\mathbf{Y}_1^M\mathbf{Z}_1^N} - p_{\mathsf{X}\mathbf{Y}_1^M\mathbf{Z}_1^N}\right\|_{\mathrm{tvd}} \leqslant \epsilon.$$

Using the data processing inequality for the total-variation distance (on the channel $(\mathsf{JKX}\mathbf{Y}_1^M\mathbf{Z}_1^N) \mapsto (\mathsf{XY_J Z_K})$) we have

$$\epsilon \geqslant \|\tilde{p}_{\mathsf{XY_JZ_K}} - p_{\mathsf{XY_JZ_K}}\|_{\mathrm{tvd}} = \left\|\tilde{p}_{\mathsf{XY_JZ_K}} - p_{\mathsf{X}} \cdot W_{\mathsf{YZ|X}}\right\|_{\mathrm{tvd}}.$$

Since the above discussion holds for all input distributions p_{X}, we have finished the proof. □

B. One-Shot Converse Bound

Lemma 3 (Special case of [10, Cor. A.14])**.** *Let $(\mathsf{X},\mathsf{Y},\mathsf{Z})$ be joint random variables distributed on $\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}$. In particular, suppose the set $\mathcal{Z}$ is finite. Then,*

$$I_{\max}(\mathsf{X}:\mathsf{YZ}) \leqslant I_{\max}(\mathsf{X}:\mathsf{Y}) + \log|\mathcal{Z}|. \tag{17}$$

Lemma 4. *Let $(\mathsf{X},\mathsf{Y},\tilde{\mathsf{Y}},\mathsf{Z},\tilde{\mathsf{Z}})$ be joint random variables distributed on $\mathcal{X}\times\mathcal{Y}\times\tilde{\mathcal{Y}}\times\mathcal{Z}\times\tilde{\mathcal{Z}}$. Suppose the set $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{Z}}$ are finite. Then,*

$$I_{\max}(\mathsf{X}:\mathsf{Y}\tilde{\mathsf{Y}}:\mathsf{Z}\tilde{\mathsf{Z}}) \leqslant I_{\max}(\mathsf{X}:\mathsf{Y}:\mathsf{Z}) + \log\left|\tilde{\mathcal{Y}}\right| + \log\left|\tilde{\mathcal{Z}}\right|. \tag{18}$$

Proof. See [9, Lemma 14]. □

Lemma 5 (Partially inspired by [11, Eq. (48)] and [12, Eq. (17)])**.** *Let $W_{\mathsf{Y|X}}$ be a DMC from $\mathcal{X}$ to $\mathcal{Y}$ where both $\mathcal{X}$ and $\mathcal{Y}$ are some finite sets. It holds for all $\epsilon \in (0,1)$ and $\delta \in (0, 1-\epsilon)$ that*

$$\begin{aligned}\inf_{\tilde{W}_{\mathsf{Y|X}}:\left\|\tilde{W}_{\mathsf{Y|X}}-W_{\mathsf{Y|X}}\right\|_{\mathrm{tvd}}\leqslant\epsilon} \max_{x\in\mathcal{X}} D_{\max}\left(\tilde{W}_{\mathsf{Y|X}}(\cdot|x)\middle\|q_{\mathsf{Y}}\right) \geqslant \\ \sup_{p_{\mathsf{X}}\in\mathcal{P}(\mathcal{X})} D_{s+}^{\epsilon+\delta}\left(p_{\mathsf{X}}\cdot W_{\mathsf{Y|X}}\middle\|p_{\mathsf{X}}\times q_{\mathsf{Y}}\right) + \log\delta\end{aligned} \tag{19}$$

for any $q_{\mathsf{Y}} \in \mathcal{P}(\mathcal{Y})$.

Proof. See [9, Lemma 16]. □

Theorem 6. *Let $W_{\mathsf{YZ|X}}$ be a DMC from $\mathcal{X}$ to $\mathcal{Y}\times\mathcal{Z}$, and let $\epsilon \in (0,1)$. For any δ_1, δ_2, $\delta_3 \in (0, 1-\epsilon)$, the following set is a superset of $\mathcal{M}_\epsilon^\star(W_{\mathsf{YZ|X}})$*

$$\mathcal{M}_\epsilon^{out}(W_{\mathsf{YZ|X}}) := \left\{(M,N)\in\mathbb{Z}_{>0}^2 : \begin{aligned} \log M &\geqslant \inf_{q_{\mathsf{Y}}}\sup_{p_{\mathsf{X}}} D_{s+}^{\epsilon+\delta_1}\left(p_{\mathsf{X}}\cdot W_{\mathsf{Y|X}}\middle\|p_{\mathsf{X}}\times q_{\mathsf{Y}}\right) + \log\delta_1 \\ \log N &\geqslant \inf_{r_{\mathsf{Z}}}\sup_{p_{\mathsf{X}}} D_{s+}^{\epsilon+\delta_2}\left(p_{\mathsf{X}}\cdot W_{\mathsf{Z|X}}\middle\|p_{\mathsf{X}}\times r_{\mathsf{Z}}\right) + \log\delta_2 \\ \log MN &\geqslant \inf_{q_{\mathsf{Y}},r_{\mathsf{Z}}}\sup_{p_{\mathsf{X}}} D_{s+}^{\epsilon+\delta_3}\left(p_{\mathsf{X}}\cdot W_{\mathsf{YZ|X}}\middle\|p_{\mathsf{X}}\times q_{\mathsf{Y}}\times r_{\mathsf{Z}}\right) + \log\delta_3\end{aligned}\right\}. \tag{20}$$

Proof. Let $(M,N) \in \mathcal{M}_\epsilon^\star(W_{\mathsf{YZ|X}})$, *i.e.*, suppose there exists a size-(M,N) ϵ-simulation code for $W_{\mathsf{YZ|X}}$. Let S and S' denote the two shared randomness between the sender and the first and the second receivers, respectively. Let M and N denote the two codewords transmitted from the sender to the first and the second receivers, respectively. Then, for any input source $\mathsf{X} \sim p_{\mathsf{X}}$, we have a Markov chain $\mathsf{Z} - \mathsf{NS}' - \mathsf{X} - \mathsf{MS} - \mathsf{Y}$ where

- X, S, and S' are independent.
- The distribution of XYZ, denoted by $\tilde{p}_{\mathsf{XYZ}}$, is ϵ-close (in TVD) to $p_{\mathsf{XYZ}} := p_{\mathsf{X}} \cdot W_{\mathsf{YZ|X}}$.
- The marginal distribution $\sum_{y,z}\tilde{p}_{\mathsf{XYZ}}(x,y,z) = p_{\mathsf{X}}(x)$ $\forall x \in \mathcal{X}$.
- M and N are distributed over $\{1,\ldots,M\}$ and $\{1,\ldots,N\}$, respectively.

Pick p_{X} to be some pmf with full support. The following statements hold.

1) By Lemma 3 and Lemma 4, we have

$$\begin{aligned}\log M &= I_{\max}(\mathsf{X}:\mathsf{S}) + \log M \\ &\geqslant I_{\max}(\mathsf{X}:\mathsf{MS}); \\ \log N &= I_{\max}(\mathsf{X}:\mathsf{S}') + \log N \\ &\geqslant I_{\max}(\mathsf{X}:\mathsf{NS}'); \\ \log M + \log N &= I_{\max}(\mathsf{X}:\mathsf{S}:\mathsf{S}') + \log M + \log N \\ &\geqslant I_{\max}(\mathsf{X}:\mathsf{MS}:\mathsf{NS}').\end{aligned}$$

2) By the data processing inequality of $I_{\max}$, *i.e.*,

$$I_{\max}(\mathsf{A}:\mathsf{B}) \geqslant I_{\max}(\mathsf{A}:\mathsf{C})$$

for any Markov chain $\mathsf{A}-\mathsf{B}-\mathsf{C}$, we have

$$\begin{aligned}I_{\max}(\mathsf{X}:\mathsf{MS}) &\geqslant I_{\max}(\mathsf{X}:\mathsf{Y}) \\ I_{\max}(\mathsf{X}:\mathsf{NS}') &\geqslant I_{\max}(\mathsf{X}:\mathsf{Z}) \\ I_{\max}(\mathsf{X}:\mathsf{MS}:\mathsf{NS}') &\geqslant I_{\max}(\mathsf{X}:\mathsf{Y}:\mathsf{Z}).\end{aligned}$$

3) By the definition of $I_{\max}$, and noting that $\tilde{p}_{\mathsf{X}} = p_{\mathsf{X}}$, we have

$$\begin{aligned}I_{\max}(\mathsf{X}:\mathsf{Y}) &= \inf_{q_Y} D_{\max}(\tilde{p}_{\mathsf{XY}}\|p_{\mathsf{X}}\times q_{\mathsf{Y}}) \\ &= \inf_{q_{\mathsf{Y}}}\max_x D_{\max}\left(\tilde{p}_{\mathsf{Y|X}}(\cdot|x)\middle\|q_{\mathsf{Y}}\right) \\ &\geqslant \inf_{q_{\mathsf{Y}}}\inf_{\left\|\tilde{W}_{\mathsf{Y|X}}-W_{\mathsf{Y|X}}\right\|_{\mathrm{tvd}}\leqslant\epsilon}\max_x D_{\max}\left(\tilde{W}_{\mathsf{Y|X}}(\cdot|x)\middle\|q_{\mathsf{Y}}\right) \\ I_{\max}(\mathsf{X}:\mathsf{Z}) &= \inf_{r_{\mathsf{Z}}} D_{\max}(\tilde{p}_{\mathsf{XZ}}\|p_{\mathsf{X}}\times r_{\mathsf{Z}}) \\ &= \inf_{r_{\mathsf{Z}}}\max_x D_{\max}\left(\tilde{p}_{\mathsf{Z|X}}(\cdot|x)\middle\|r_{\mathsf{Z}}\right) \\ &\geqslant \inf_{r_{\mathsf{Z}}}\inf_{\left\|\tilde{W}_{\mathsf{Z|X}}-W_{\mathsf{Z|X}}\right\|_{\mathrm{tvd}}\leqslant\epsilon}\max_x D_{\max}\left(\tilde{W}_{\mathsf{Z|X}}(\cdot|x)\middle\|r_{\mathsf{Z}}\right) \\ I_{\max}(\mathsf{X}:\mathsf{Y}:\mathsf{Z}) &= \inf_{q_{\mathsf{Y}}}\inf_{r_{\mathsf{Z}}} D_{\max}(\tilde{p}_{\mathsf{XYZ}}\|p_{\mathsf{X}}\times q_{\mathsf{Y}}\times r_{\mathsf{Z}}) \\ &= \inf_{q_{\mathsf{Y}}}\inf_{r_{\mathsf{Z}}}\max_x D_{\max}\left(\tilde{p}_{\mathsf{YZ|X}}(\cdot|x)\middle\|q_{\mathsf{Y}}\times r_{\mathsf{Z}}\right) \\ &\geqslant \inf_{q_{\mathsf{Y}}}\inf_{r_{\mathsf{Z}}}\inf_{\left\|\tilde{W}_{\mathsf{YZ|X}}-W_{\mathsf{YZ|X}}\right\|_{\mathrm{tvd}}\leqslant\epsilon}\max_x D_{\max}\left(\tilde{W}_{\mathsf{YZ|X}}(\cdot|x)\middle\|q_{\mathsf{Y}}\times r_{\mathsf{Z}}\right).\end{aligned}$$

The theorem can be proven by combining the above three steps and Lemma 5. □

III. Simulation Rate Region of Broadcast Channels

In this section, we consider the task of simulating $W_{\mathsf{YZ|X}}^{\otimes n}$. In asymptotic discussions, one is usually more interested in admissible rates instead of admissible messages sizes. For our

$$\sup_{\boldsymbol{x}_1^n}\left\{\frac{1}{n}\sum_{i=1}^n D\left(W_{\mathsf{YZ|X}}(\cdot|x_i)\middle\|\left(\sum_{\tilde{x},\tilde{z}}W_{\mathsf{YZ|X}}(\cdot,\tilde{z}|\tilde{x})\cdot f_{\boldsymbol{x}_1^n}(\tilde{x})\right)\times\left(\sum_{\tilde{x},\tilde{y}}W_{\mathsf{YZ|X}}(\tilde{y},\cdot|\tilde{x})\cdot f_{\boldsymbol{x}_1^n}(\tilde{x})\right)\right)\right.$$

$$\left.+\frac{1}{\sqrt{n(\epsilon_3-\delta_3)}}\cdot\sqrt{\frac{1}{n}\sum_{i=1}^n V\left(W_{\mathsf{YZ|X}}(\cdot|x_i)\middle\|\left(\sum_{\tilde{x},\tilde{z}}W_{\mathsf{YZ|X}}(\cdot,\tilde{z}|\tilde{x})\cdot f_{\boldsymbol{x}_1^n}(\tilde{x})\right)\times\left(\sum_{\tilde{x},\tilde{y}}W_{\mathsf{YZ|X}}(\tilde{y},\cdot|\tilde{x})\cdot f_{\boldsymbol{x}_1^n}(\tilde{x})\right)\right)}\right\} \quad (*)$$

$$-\frac{1}{n}\log\delta_3^2+\frac{2}{n}\log|\Lambda_n|$$

task of simulating $W_{\mathsf{YZ|X}}$ asymptotically, a rate pair (r_1, r_2) (of positive real numbers) is said to be ϵ-attainable if there exists a sequence of size-$(\lfloor 2^{nr_1}\rfloor, \lfloor 2^{nr_2}\rfloor)$ ϵ-simulation codes for $W_{\mathsf{YZ|X}}^{\otimes n}$ for n sufficiently large. We denote $\mathcal{R}_\epsilon^\star(W_{\mathsf{YZ|X}})$ the *closure* of the set of all ϵ-attainable rate pairs, *i.e.*,

$$\mathcal{R}_\epsilon^\star(W_{\mathsf{YZ|X}}) \coloneqq \mathrm{cl}\Big(\Big\{(r_1,r_2)\in\mathbb{R}^2_{\geqslant 0}\Big|\exists N\in\mathbb{N} \text{ s.t. } (\lfloor 2^{nr_1}\rfloor, \lfloor 2^{nr_2}\rfloor)\in\mathcal{M}_\epsilon^\star(W_{\mathsf{YZ|X}}^{\otimes n}) \text{ for all } n\geqslant N\Big\}\Big) \quad (21)$$

Theorem 7. *Let $W_{\mathsf{YZ|X}}$ be a DMC from $\mathcal{X}$ to $\mathcal{Y}\times\mathcal{Z}$, and let $\epsilon\in(0,1)$. It holds that*

$$\mathcal{R}_\epsilon^\star(W_{\mathsf{YZ|X}}) = \left\{(r_1,r_2)\in\mathbb{R}^2_{\geqslant 0}\,\middle|\,\begin{array}{r} r_1\geqslant C(W_{\mathsf{Y|X}})\\ r_2\geqslant C(W_{\mathsf{Z|X}})\\ r_1+r_2\geqslant \tilde{C}(W_{\mathsf{YZ|X}})\end{array}\right\}. \quad (22)$$

We need the following lemma to prove the above theorem.

Lemma 8. *For any $p_{\mathsf{XY}}\in\mathcal{P}(\mathcal{X}\times\mathcal{Y}\times\mathcal{Z})$, $\epsilon\in(0,1)$, and $\delta\in(0,\frac{1-\epsilon}{2})$, it holds that*

$$\inf_{q_{\mathsf{Y}}\in\mathcal{P}(\mathcal{Y}), r_{\mathsf{Z}}\in\mathcal{P}(\mathcal{Z})} D_{s+}^{\epsilon}(p_{\mathsf{XYZ}}\|p_{\mathsf{X}}\times q_{\mathsf{Y}}\times r_{\mathsf{Z}}) \geqslant D_{s+}^{\epsilon+2\delta}(p_{\mathsf{XY}}\|p_{\mathsf{X}}\times p_{\mathsf{Y}}\times p_{\mathsf{Z}})+2\log\delta. \quad (23)$$

Proof. See [9, Lemma 18]. □

A. Achievability Proof of Theorem 7

Applying Theorem 2, we have the following set being a subset of $\mathcal{M}_\epsilon^\star(W_{\mathsf{YZ|X}}^{\otimes n})$

$$\mathcal{M}_\epsilon^{\text{in}}(W_{\mathsf{YZ|X}}^{\otimes n}) \coloneqq \left\{(M,N)\in\mathbb{Z}^2_{>0}: \begin{array}{r}\log M > D_{s+}^{\epsilon_1-\delta_1}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{Y|X}}^{\otimes n}\middle\|p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\right)-\log\delta_1^2\\ \log N > D_{s+}^{\epsilon_2-\delta_2}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{Z|X}}^{\otimes n}\middle\|p_{\mathsf{X}_1^n}\times r_{\mathsf{Z}_1^n}\right)-\log\delta_2^2\\ \log MN > D_{s+}^{\epsilon_3-\delta_3}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{YZ|X}}^{\otimes n}\middle\|p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\times r_{\mathsf{Z}_1^n}\right)-\log\delta_3^2\\ \forall p_{\mathsf{X}_1^n}\in\mathcal{P}(\mathcal{X}^n)\end{array}\right\}.$$

for any $\epsilon_1,\epsilon_2,\epsilon_3>0$ such that $\epsilon_1+\epsilon_2+\epsilon_3\leqslant\epsilon$, $\delta_1\in(0,\epsilon_1)$, $\delta_2\in(0,\epsilon_2)$, $\delta_3\in(0,\epsilon_3)$, $q_{\mathsf{Y}_1^n}\in\mathcal{P}(\mathcal{Y}^n)$, and $r_{\mathsf{Z}_1^n}\in\mathcal{P}(\mathcal{Z}^n)$. We pick $q_{\mathsf{Y}_1^n}$ and $r_{\mathsf{Z}_1^n}$ as

$$q_{\mathsf{Y}_1^n} \coloneqq \sum_{\lambda\in\Lambda_n}\frac{1}{|\Lambda_n|}\left(\sum_{\tilde{x},\tilde{z}}W_{\mathsf{YZ|X}}(\cdot,\tilde{z}|\tilde{x})\cdot p_\lambda(\tilde{x})\right)^{\otimes n},$$

$$r_{\mathsf{Z}_1^n} \coloneqq \sum_{\lambda\in\Lambda_n}\frac{1}{|\Lambda_n|}\left(\sum_{\tilde{x},\tilde{y}}W_{\mathsf{YZ|X}}(\tilde{y},\cdot|\tilde{x})\cdot p_\lambda(\tilde{x})\right)^{\otimes n}.$$

We have the following chain of inequalities.

$$\inf_{q_{\mathsf{Y}_1^n}, r_{\mathsf{Z}_1^n}}\sup_{p_{\mathsf{X}_1^n}}\frac{1}{n}D_{s+}^{\epsilon_3-\delta_3}\Big(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{YZ|X}}^{\otimes n}\Big\|p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\times r_{\mathsf{Z}_1^n}\Big)-\frac{1}{n}\log\delta_3^2$$

$$\overset{a)}{\leqslant}\inf_{q_{\mathsf{Y}_1^n}, r_{\mathsf{Z}_1^n}}\sup_{\boldsymbol{x}_1^n}\frac{1}{n}D_{s+}^{\epsilon_3-\delta_3}\Big(W_{\mathsf{YZ|X}}^{\otimes n}(\cdot|\boldsymbol{x}_1^n)\Big\|q_{\mathsf{Y}_1^n}\times r_{\mathsf{Z}_1^n}\Big)-\frac{1}{n}\log\delta_3^2$$

$$\overset{b)}{\leqslant}\sup_{\boldsymbol{x}_1^n}\frac{1}{n}D_{s+}^{\epsilon_3-\delta_3}\left(W_{\mathsf{YZ|X}}^{\otimes n}(\cdot|\boldsymbol{x}_1^n)\middle\|\sum_{\lambda\in\Lambda_n}\frac{1}{|\Lambda_n|}\left(\sum_{\tilde{x},\tilde{z}}W_{\mathsf{YZ|X}}(\cdot,\tilde{z}|\tilde{x})\cdot p_\lambda(\tilde{x})\right)^{\otimes n}\times\sum_{\lambda\in\Lambda_n}\frac{1}{|\Lambda_n|}\left(\sum_{\tilde{x},\tilde{y}}W_{\mathsf{YZ|X}}(\tilde{y},\cdot|\tilde{x})\cdot p_\lambda(\tilde{x})\right)^{\otimes n}\right)-\frac{1}{n}\log\delta_3^2$$

$$\overset{c)}{\leqslant}\sup_{\boldsymbol{x}_1^n}\frac{1}{n}D_{s+}^{\epsilon_3-\delta_3}\left(W_{\mathsf{YZ|X}}^{\otimes n}(\cdot|\boldsymbol{x}_1^n)\middle\|\left(\sum_{\tilde{x},\tilde{z}}W_{\mathsf{YZ|X}}(\cdot,\tilde{z}|\tilde{x})\cdot f_{\boldsymbol{x}_1^n}(\tilde{x})\right)^{\otimes n}\times\left(\sum_{\tilde{x},\tilde{y}}W_{\mathsf{YZ|X}}(\tilde{y},\cdot|\tilde{x})\cdot f_{\boldsymbol{x}_1^n}(\tilde{x})\right)^{\otimes n}\right)-\frac{1}{n}\log\delta_3^2+\frac{2}{n}\log|\Lambda_n|$$

$\overset{d)}{\leqslant}$ $(*)$ at the top of this page

$$\overset{e)}{\leqslant}\sup_{p_{\mathsf{X}}}I(\mathsf{X}:\mathsf{Y}:\mathsf{Z})_{p_{\mathsf{X}}\cdot W_{\mathsf{YZ|X}}}+\frac{1}{\sqrt{n(\epsilon_3-\delta_3)}}\cdot\tilde{V}(p_{\mathsf{X}})-\frac{1}{n}\log\delta_3^2+\frac{2}{n}\log|\Lambda_n|$$

where for a) we use the quasi-convexity of $D_{s+}(p_{\mathsf{X}}\cdot p_{\mathsf{Y|X}}\|p_{\mathsf{X}}\cdot q_{\mathsf{Y|X}})$ in p_{X}, for b) we pick a pair of specific $q_{\mathsf{Y}_1^n}$ and $r_{\mathsf{Z}_1^n}$ as aforementioned to upper bound the infimum, for c) we use [13, Lemma 3], for d) we use the Chebyshev-type bound in [13, Lemma 5], and e) is a result of direct counting and the definition of the *common dispersion* $\tilde{V}$ as

$$\tilde{V}(p)\coloneqq\sum_x p(x)\cdot V\left(W_{\mathsf{YZ|X}}(\cdot,\cdot|x)\middle\|\sum_{\tilde{x},\tilde{z}}p(x)\cdot W_{\mathsf{YZ|X}}(\cdot,\tilde{z}|\tilde{x})\times\sum_{\tilde{x},\tilde{y}}p(x)\cdot W_{\mathsf{YZ|X}}(\tilde{y},\cdot|\tilde{x})\right).$$

Notice that $\tilde{V}$ is bounded. Thus, it holds that

$$\limsup_{n\to\infty} \inf_{q_{\mathsf{Y}_1^n}, r_{\mathsf{Z}_1^n}} \sup_{p_{\mathsf{X}_1^n}} \frac{1}{n} D_{s+}^{\epsilon_3-\delta_3}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\times r_{\mathsf{Z}_1^n}\right) - \frac{1}{n}\log\delta_3^2 \leqslant \sup_{p_\mathsf{X}} I(\mathsf{X}:\mathsf{Y}:\mathsf{Z})_{p_\mathsf{X}\cdot W_{\mathsf{YZ}|\mathsf{X}}} = \tilde{C}(W_{\mathsf{YZ}|\mathsf{X}}). \quad \text{(A)}$$

Similarly, one can show

$$\limsup_{n\to\infty} \inf_{q_{\mathsf{Y}_1^n}} \sup_{p_{\mathsf{X}_1^n}} \frac{1}{n} D_{s+}^{\epsilon_1-\delta_1}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{Y}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\right) - \frac{1}{n}\log\delta_1^2 \leqslant C(W_{\mathsf{Y}|\mathsf{X}}), \quad \text{(B)}$$

$$\limsup_{n\to\infty} \inf_{r_{\mathsf{Z}_1^n}} \sup_{p_{\mathsf{X}_1^n}} \frac{1}{n} D_{s+}^{\epsilon_2-\delta_2}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{Z}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times r_{\mathsf{Z}_1^n}\right) - \frac{1}{n}\log\delta_2^2 \leqslant C(W_{\mathsf{Z}|\mathsf{X}}). \quad \text{(C)}$$

Combining (A), (B), and (C) with the expression of $\mathcal{M}_\epsilon^{\text{in}}(W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n})$ at the begining of this proof, it is straightforward to check that any integer pair $(\lfloor 2^{nr_1}\rfloor, \lfloor 2^{nr_2}\rfloor)$ with

$$(r_1,r_2)\in\left\{(r_1,r_2)\in\mathbb{R}_{\geqslant 0}^2 \middle| \begin{array}{r} r_1 > C(W_{\mathsf{Y}|\mathsf{X}}) \\ r_2 > C(W_{\mathsf{Z}|\mathsf{X}}) \\ r_1+r_2 > \tilde{C}(W_{\mathsf{YZ}|\mathsf{X}}) \end{array}\right\}, \quad \text{(D)}$$

must be in $\mathcal{M}_\epsilon^{\text{in}}(W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n})$ for n sufficiently large, *i.e.*, RHS of (D)$\subset \mathcal{R}_\epsilon^\star(W_{\mathsf{YZ}|\mathsf{X}})$. This proves the RHS of (22) being a subset of $\mathcal{R}_\epsilon^\star(W_{\mathsf{YZ}|\mathsf{X}})$ since the latter is a closed set.

B. Converse Proof of Theorem 7

Let (r_1,r_2) be arbitrarily pair of non-negative numbers such that $(2^{\lfloor nr_1\rfloor}, 2^{\lfloor nr_2\rfloor}) \in \mathcal{M}_\epsilon^\star(W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n})$ for n sufficiently large, *i.e.*, (r_1,r_2) is an arbitrarily *interior* point of $\mathcal{R}_\epsilon^\star(W_{\mathsf{YZ}|\mathsf{X}})$. Applying Theorem 6, we have the following set being a superset of $\mathcal{M}_\epsilon^\star(W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n})$

$$\mathcal{M}_\epsilon^{\text{out}}(W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n}) \coloneqq \left\{(M,N)\in\mathbb{Z}_{>0}^2 : \begin{array}{l} \log M \geqslant \inf_{q_{\mathsf{Y}_1^n}} \sup_{p_{\mathsf{X}_1^n}} D_{s+}^{\epsilon+\delta_1}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{Y}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\right) + \log\delta_1 \\ \log N \geqslant \inf_{r_{\mathsf{Z}_1^n}} \sup_{p_{\mathsf{X}_1^n}} D_{s+}^{\epsilon+\delta_2}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{Z}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times r_{\mathsf{Z}_1^n}\right) + \log\delta_2 \\ \log MN \geqslant \inf_{q_{\mathsf{Y}_1^n}, r_{\mathsf{Z}_1^n}} \sup_{p_{\mathsf{X}_1^n}} D_{s+}^{\epsilon+\delta_3}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\times r_{\mathsf{Z}_1^n}\right) + \log\delta_3 \end{array}\right\}.$$

for any δ_1, δ_2, $\delta_3 \in (0, 1-\epsilon)$. Thus, we have

$$r_1 \overset{\text{a)}}{\geqslant} \inf_{q_{\mathsf{Y}_1^n}} \sup_{p_{\mathsf{X}_1^n}} \frac{1}{n} D_{s+}^{\epsilon+\delta_1}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{Y}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\right) + \frac{1}{n}\log\delta_1$$

$$r_2 \overset{\text{b)}}{\geqslant} \inf_{r_{\mathsf{Z}_1^n}} \sup_{p_{\mathsf{X}_1^n}} \frac{1}{n} D_{s+}^{\epsilon+\delta_2}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{Z}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times r_{\mathsf{Z}_1^n}\right) + \frac{1}{n}\log\delta_2$$

$$r_1+r_2 \overset{\text{c)}}{\geqslant} \inf_{q_{\mathsf{Y}_1^n}, r_{\mathsf{Z}_1^n}} \sup_{p_{\mathsf{X}_1^n}} \frac{1}{n} D_{s+}^{\epsilon+\delta_3}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\times r_{\mathsf{Z}_1^n}\right) + \frac{1}{n}\log\delta_3$$

for n sufficiently large. By Lemma 8, we can rewrite c) as

$$\begin{aligned} r_1+r_2 &\geqslant \sup_{p_{\mathsf{X}_1^n}} \inf_{q_{\mathsf{Y}_1^n}, r_{\mathsf{Z}_1^n}} \frac{1}{n} D_{s+}^{\epsilon+\delta_3}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times q_{\mathsf{Y}_1^n}\times r_{\mathsf{Z}_1^n}\right) + \frac{1}{n}\log\delta_3 \\ &\geqslant \sup_{p_{\mathsf{X}_1^n}} \frac{1}{n} D_{s+}^{\epsilon+3\delta_3}\left(p_{\mathsf{X}_1^n}\cdot W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n} \middle\| p_{\mathsf{X}_1^n}\times p_{\mathsf{Y}_1^n}\times p_{\mathsf{Z}_1^n}\right) + \frac{3}{n}\log\delta_3. \quad \text{(E)} \end{aligned}$$

Using the information spectrum method [14], we know $\lim_{n\to\infty}$ RHS of (E) $= \tilde{C}(W_{\mathsf{YZ}|\mathsf{X}})$. Since (E) holds for all n sufficiently large, the inequality is maintained as $n\to\infty$, *i.e.*, $r_1+r_2 \geqslant \tilde{C}(W_{\mathsf{YZ}|\mathsf{X}})$.

Similarly, using [11, Lemma 10] with a) and b), one can show $r_1 \geqslant C(W_{\mathsf{Y}|\mathsf{X}})$ and $r_2 \geqslant C(W_{\mathsf{Z}|\mathsf{X}})$, respectively.

Since (r_1,r_2) are picked arbitrarily, we have shown

$$\left\{(r_1,r_2)\in\mathbb{R}_{\geqslant 0}^2 \middle| \exists N\in\mathbb{N} \text{ s.t. } (\lfloor 2^{nr_1}\rfloor, \lfloor 2^{nr_2}\rfloor)\in\mathcal{M}_\epsilon^\star(W_{\mathsf{YZ}|\mathsf{X}}^{\otimes n}) \text{ for all } n\geqslant N\right\} \subset \text{LHS of (22)}.$$

Finally, taking closure of the sets on both sides we have $\mathcal{R}_\epsilon^\star(W_{\mathsf{YZ}|\mathsf{X}}) \subset$ LHS of (22).

C. Numerical Example

The mutual information and multi-partite common information can be computed via a Blahut–Arimoto type algorithm [15], [16]. We refer to [9, Section V-D] for details.

IV. Conclusion

In this paper, we investigated the task of simulating discrete broadcast channel in both one-shot and i.i.d. setups. The one-shot results are based on the bipartite convex split lemma and multi-partite generalizations of some of the popular tools in the finite blocklength information theory. Based on the one-shot results, we managed to obtain a single-letter expression of the simulation region of the broadcast channels, which is very different from the corresponding channel coding problem. We would like to point out that all work presented in this paper can be generalized to K-receiver broadcast channel rather straightforwardly.

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