Open question:
Is there a way of making entanglement theory reversible?

On a gap in the proof of the generalised quantum Stein's lemma
and its consequences for the reversibility of quantum resources

B., Brandão, Gour, Lami, Plenio, Regula, Tomamichel
arXiv:2205.02813 (2022)

Resource theory of entanglement

- Free states are given by convex hull of product states

$$S_{A:B} \coloneqq conv\{|\psi_A\rangle\langle\psi_A| \otimes |\phi\rangle\langle\phi|_B : |\psi\rangle_A \in H_A, |\phi\rangle_B \in H_B\}$$

- Measure global resource robustness $R_S(\rho_{AB}) \coloneqq \{s \geq 0: \frac{\rho_{AB}+s\sigma_{AB}}{1+s} \in S_{A:B}\}$
- Largest set of free operations for transformations $\rho_{AB} \to \omega_{AB}$ are δ-non-entangling operations (ANE)

$$NE_{\delta(A:B\to A':B')} \coloneqq \{\Lambda \in CPTP(AB \to A'B'): R_S\big(\Lambda(\sigma_{AB})\big) \leq \delta\ \forall\sigma_{AB} \in S_{A:B}\}$$

- Distillable entanglement / entanglement cost under ANE (asymptotic rates)
- Open question: Do we have the reversibility $E_D^{ANE}(\rho) = E_C^{ANE}(\rho)$?

Asymptotic characterization of entanglement

- Entanglement cost [Brandão & Plenio, CMP 10], [Datta, IEEE 09]

$$E_C^{ANE}(\rho) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\min_{\sigma^n\in S^n} D_{\max}^{\varepsilon}(\rho^{\otimes n}||\sigma^n) = \lim_{n\to\infty}\frac{1}{n}\min_{\sigma^n\in S^n} D(\rho^{\otimes n}||\sigma^n) \neq \min_{\sigma\in S} D(\rho||\sigma)$$

- Distillable entanglement [Brandão & Plenio, CMP 10]

$$E_D^{ANE}(\rho) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} -\frac{1}{n}\log\beta_\varepsilon\left(\rho^{\otimes n}, S^n\right)$$

for the hypothesis testing $\beta_\varepsilon\left(\rho^{\otimes n}, S^n\right) \coloneqq \inf_{0\le M_n\le 1}\{\sup_{\sigma^n\in S^n} Tr[M_n\sigma^n]\colon Tr\left[(1-M_n)\rho^{\otimes n}\right]\le\varepsilon\}$

- Composite quantum hypothesis testing question $-\frac{1}{n}\log\beta_\varepsilon\left(\rho^{\otimes n}, S^n\right) \to \frac{1}{n}\min_{\sigma^n\in S^n} D(\rho^{\otimes n}||\sigma^n)$?

Reduction to hypothesis testing

- Question if $E_D^{ANE}(\rho) = E_C^{ANE}(\rho)$ reduces to composite quantum hypothesis question

$$-\frac{1}{n}\log \beta_\varepsilon(\rho, S^n) \to \frac{1}{n}\min_{\sigma^n \in S^n} D(\rho^{\otimes n}||\sigma^n) \ ?$$

- Converse direction by standard arguments [Brandão & Plenio, CMP 10]

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} -\frac{1}{n}\log \beta_\varepsilon(\rho, S^n) \leq \lim_{n\to\infty}\frac{1}{n}\min_{\sigma^n \in S^n} D(\rho^{\otimes n}||\sigma^n)$$

- [B. *et al.*, arXiv 22] recently found that achievability direction " $\geq$ " remains open

- Needed: Composite hypothesis test that asymptotically achieves regularized relative entropy of entanglement!

Known proof techniques

I. via Petz-Rényi divergences

II. via measured divergence

III. via smooth max-relative entropy

IV. find counterexamples to conjecture

Some starting references (incomplete)

- B., Brandão, Hirche: CMP 385, 55 (2021)
- B., Brandão, Gour, Lami, Plenio, Regula, Tomamichel: arXiv:2205.02813 (2022)
- Audenaert, Nussbaum, Szkola, Verstraete: CMP 279, 251 (2008)
- Brandão, Harrow, Lee, Peres: IEEE Trans. 66, 5037 (2020)
- Mosonyi, Szilágyi, Weiner: arXiv:2011.04645 (2021)
- Lami & Regula: arXiv:2111.02438 (2021)
- Bergh, Datta, Salzmann, Wilde: arXiv:2206.08350 (2022)

I. Universal hypothesis tests via Petz-Rényi divergences

- Sion minimax + Audenaert inequality for $s \in (0,1)$ gives [Audenaert *et al.*, CMP 08]

$$-\frac{1}{n}\log\beta_\varepsilon\left(\rho^{\otimes n}, S^n\right) = -\frac{1}{n}\sup_{\sigma^n\in S^n}\inf_{0\le M_n\le 1, Tr\left[M_n\rho^{\otimes n}\right]\ge 1-\varepsilon}\log Tr[M_n\sigma^n] \ge \frac{1}{n}\inf_{\sigma^n\in S^n} D_s(\rho^{\otimes n}||\sigma^n) - \frac{1}{n}\cdot\frac{s}{1-s}\log\frac{1}{\varepsilon}$$

for the additive $D_s(\rho||\sigma) := \frac{1}{s-1}\log Tr[\rho^s\sigma^{1-s}]$ with $\lim_{s\to 1} \mathrm{D_s}(\rho||\sigma) = D(\rho||\sigma)$

- Single-letter: de Finetti, take limits $(\mathrm{i})\ n\to\infty\ (ii)\ \varepsilon\to 0\ (iii)\ s\to 1$ in order [B. et al., CMP 21]
- Generally, with information variance $V(\rho||\sigma) := Tr[\rho(\log\rho - \log\sigma - D(\rho||\sigma))^2$) to bound

$$\frac{1}{n}|D_s(\rho^{\otimes n}||\sigma^n) - D(\rho^{\otimes n}||\sigma^n)| \le \frac{s-1}{2}\cdot\frac{V(\rho^{\otimes n}||\sigma^n)}{n} + \frac{O\left((s-1)^2\right)}{n}$$

- [Brandão & Plenio, CMP 10] claimed that $V(\rho^{\otimes n}||\sigma^n) \le o(2^{-n})$, but already

$V(\rho^{\otimes n}||\sigma^{\otimes n}) = n\cdot V(\rho||\sigma) \nleq o(2^{-n})$ → Remains open: de Finetti / Schur-Weyl duality?

II. Universal hypothesis tests via measured divergence

- Measured relative entropy [Donald, CMP 86] with [Brandão *et al.*, IEEE 20]

$$D_M(\rho||\sigma) \coloneqq \sup_M D_{KL}(M(\rho)\big|\big|M(\sigma)\big) \text{ with } \inf_{\rho\in T,\sigma\in S} D_M(\rho||\sigma) = \sup_M \inf_{\rho\in T,\sigma\in S} D_{KL}(M(\rho)||M(\sigma))$$

- (i) measure, (ii) apply classical composite hypothesis result, (iii) use asymptotic achievability of measured relative entropy for ρ^n, σ^n permutation invariant [B. et al., CMP 21]

$$\frac{1}{n} D_M(\rho^n||\sigma^n) \to \frac{1}{n} D(\rho^n||\sigma^n) \text{ for } n \to \infty$$

- Gives pseudo-entanglement theory and pseudo-entanglement in blocks [B. *et al.*, arXiv 22]
- Entanglement theory remains open!
- Alternatively, one has [Brandão *et al.*, IEEE 20]

$$\lim_{n\to\infty} \frac{1}{n} \min_{\sigma^n\in S^n} D_{M_{SEP}}(\rho^{\otimes n}||\sigma^n) \to \lim_{n\to\infty} \frac{1}{n} \min_{\sigma^n\in S^n} D(\rho^{\otimes n}||\sigma^n)\,?$$

III. Universal hypothesis tests via max-relative entropy

- For $\varepsilon \in (0,1)$ we have [Anshu *et al.*, JMP 19]

$$-\frac{1}{n}\log \sup_{\sigma^n \in S^n} \beta_\varepsilon\left(\rho^{\otimes n}, \sigma^n\right) \geq \frac{1}{n} \min_{\sigma^n \in S^n} D_{\max}^{\sqrt{1-\varepsilon}}(\rho^{\otimes n} || \sigma^n) - \frac{1}{n}\log\frac{1}{\varepsilon}$$

- Previously mentioned asymptotic equipartition property (AEP) for max-relative entropy

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\min_{\sigma^n \in S^n} D_{\max}^{\delta}(\rho^{\otimes n}||\sigma^n) = \lim_{n\to\infty}\frac{1}{n}\min_{\sigma^n\in S^n} D(\rho^{\otimes n}||\sigma^n)$$ not enough as no strong converse!

- Similar open problems:
  - Quantum channel AEP [Gour & Winter, PRL 19]
  - Strong converse channel discrimination & channel capacities [Fang *et al.*, arXiv 21] [Bergh *et al.*, arXiv 21]
  - Stronger entropy accumulation [Metger *et al.*, arXiv 22]

IV. Lami & Regula arXiv:2111.02438

- Title: No second law of entanglement manipulation after all
- Recall:
  - δ-non-entangling operations $NE_{\delta(AB\to A'B')} = \{\Lambda \in CPTP(AB \to A'B'): R_S\big(\Lambda(\sigma_{AB})\big) \leq \delta \ \forall \sigma_{AB} \in S_{A:B}\}$
  - with global resource robustness $R_S(\rho_{AB}) = \{s \geq 0: \frac{\rho_{AB}+s\sigma_{AB}}{1+s} \in S_{A:B}\}$
- Replace $R_S(\rho_{AB})$ with resource robustness

$$\bar{R}_s(\rho_{AB}) := \{s \geq 0: \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}, \sigma_{AB} \in S_{A:B}\} \geq R_S(\rho_{AB})$$

and correspondingly

$$\overline{NE}_{\delta(AB\to A'B')} \coloneqq \{\Lambda \in CPTP(AB \to A'B'): \bar{R}_S\big(\Lambda(\sigma_{AB})\big) \leq \delta \ \forall \sigma_{AB} \in S_{AB}\}$$

- Main result: there exists quantum state ρ with $E_D^{\overline{ANE}}(\rho) < E_C^{\overline{ANE}}(\rho)$

Is there a way of making entanglement theory reversible?

→ Becomes composite quantum hypothesis testing question

$$-\frac{1}{n}\log \beta_\varepsilon(\rho^{\otimes n}, S^n) \geq \frac{1}{n}\min_{\sigma^n \in S^n} D(\rho^{\otimes n}||\sigma^n)?$$

with $\beta_\varepsilon(\rho^{\otimes n}, S^n) \coloneqq \inf_{0\leq M_n \leq 1}\{\sup_{\sigma^n \in S^n} Tr[M_n\sigma^n]\colon Tr[(1-M_n)\rho^{\otimes n}] \leq \varepsilon\}$

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