Quantum Coding via Semidefinite Programming

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***Abstract*—We derive converging hierarchies of efficiently computable semidefinite programming outer bounds on the optimal fidelity for the transmission of quantum information over noisy quantum channels. Based on positive partial transpose conditions we give a sufficient criterion for the exact convergence at any given level of the hierarchies. The worst case convergence speed of our hierarchies is quantified via positive semidefinite representable outer approximations on the set of separable Choi states, which are based on novel finite de Finetti theorems for quantum channels.**

I. Introduction

Given a noisy classical channel $N_{X\to Y}$, a central quantity of interest in error correction is the maximum success probability $p(N,M)$ for transmitting a uniform M-dimensional message under the noise model $N_{X\to Y}$. This is a bilinear maximization problem, which is in general NP-hard to approximate up to a sufficiently small constant factor [1]. Nevertheless, there are efficient methods for constructing feasible coding schemes approximating $p(N,M)$ from below as well as an efficiently computable linear programming relaxation $\mathrm{lp}(N,M)$ (sometimes called meta converse [2], [3]) giving upper bounds on $p(N,M)$. In fact, it was shown in [1] that $p(N,M)$ and $\mathrm{lp}(N,M)$ cannot be very far from each other

$$p(N,M)\leq \mathrm{lp}(N,M)\leq \frac{1}{1-\frac{1}{e}}\cdot p(N,M).$$

Furthermore, the meta-converse has many appealing analytic properties, such as, e.g., the asymptotic expansion in the limit of many independent repetitions $N_{X\to Y}^{\times n}$, leading to very precise asymptotic bounds on the capacity of noisy classical channels.

The analogue quantum problem is to determine the channel fidelity $F(\mathcal{N},M)$ for transmitting one part of a maximally entangled state of dimension M over a noisy quantum channel $\mathcal{N}_{A\to B}$. As in the classical case, this is a bilinear optimization problem, only now with matrix-valued variables. In order to approximate $F(\mathcal{N},M)$, an efficiently computable semidefinite programming relaxation $\mathrm{sdp}(\mathcal{N},M)$ was given in [4]. However, contrary to the classical case the gap between $\mathrm{sdp}(\mathcal{N},M)$ and $F(\mathcal{N},M)$ is not understood. Moreover, the relaxation $\mathrm{sdp}(\mathcal{N},M)$ is lacking most of the analytic properties of its classical analogue $\mathrm{lp}(N,M)$, such as, e.g., the non-accessible asymptotic expansion in the limit of many independent repetitions $\mathcal{N}_{A\to B}^{\otimes n}$.

Numerical lower bound methods for $F(\mathcal{N},M)$ are available through iterative seesaw methods that lead to efficiently computable semidefinite programs [5]–[8]. These algorithms often converge in practice and sometimes even provably reach a local maximum. What is missing, however, is a general method to give an approximation guarantee to the global maximum. In this paper, we develop techniques that lead to a converging hierarchy of efficiently computable semidefinite programming relaxations on the channel fidelity $F(\mathcal{N},M)$. This can be seen as a tool for benchmarking existing quantum codes and to understand in what direction to look for improved codes.

II. Quantum coding

The quantum version of the average error probability criterion for fixed message dimension M is as follows.

Definition II.1. *Let $\mathcal{N}_{\bar{A}\to B}$ be a quantum channel and $M\in\mathbb{N}$. The channel fidelity is defined as*

$$\begin{aligned}&F(\mathcal{N},M)\\&:=\max F\Big(\Phi_{\bar{B}R},\left(\left(\mathcal{D}_{B\to\bar{B}}\circ\mathcal{N}_{\bar{A}\to B}\circ\mathcal{E}_{A\to\bar{A}}\right)\otimes\mathcal{I}_R\right)(\Phi_{AR})\Big)\\&\qquad\text{s.t. } \mathcal{D}_{B\to\bar{B}},\mathcal{E}_{A\to\bar{A}} \text{ quantum channels}\end{aligned}$$

where $F(\rho,\sigma):=\left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1^2$ denotes the fidelity, $\mathcal{I}_R$ the identity channel on R, Φ_{AR} the maximally entangled state on AR, and we have $M=d_A=d_{\bar{B}}=d_R$.

This is conveniently rewritten as a matrix-valued bilinear optimization by employing the Choi isomorphism $J(\mathcal{M})_{AB}:=(\mathcal{I}_A\otimes\mathcal{M}_{A\to B})(\Phi_{AA})$.

Lemma II.2. *Let $\mathcal{N}_{\bar{A}\to B}$ be a quantum channel and $M\in\mathbb{N}$. Then, we have that*

$$\begin{aligned}&F(\mathcal{N},M)\\&=d_{\bar{A}}d_B\cdot\max\operatorname{Tr}\left[\left(J(\mathcal{N})_{\bar{A}B}\otimes\Phi_{A\bar{B}}\right)\left(E_{A\bar{A}}\otimes D_{B\bar{B}}\right)\right]\\&\qquad\text{s.t. } E_{A\bar{A}}\succeq 0,\ E_A=\frac{1_A}{d_A}\\&\qquad\quad D_{B\bar{B}}\succeq 0,\ D_B=\frac{1_B}{d_B}\end{aligned}$$

where $\succeq$ denotes the positive semidefinite Loewner order.

Proof. See the full version [9, Lem. 5.2]. □

Note that in the objective function the cut for the two tensor products is not the same: the encoder-decoder pair is separable between the A- and B-systems, whereas the other operator is entangled in this cut. Alternatively, we can also work with the following maximum error probability criterion.

Definition II.3. *Let $\mathcal{N}_{\bar{A}\to B}$ be a quantum channel and $M \in \mathbb{N}$. The channel distance is defined as*

$$\Delta(\mathcal{N},\mathcal{I}_M) := \min \frac{1}{2}\left\|\mathcal{D}_{B\to\bar{B}} \circ \mathcal{N}_{\bar{A}\to B} \circ \mathcal{E}_{A\to\bar{A}} - \mathcal{I}_{A\to\bar{B}}\right\|_\diamond \quad \text{s.t. } \mathcal{D}_{B\to\bar{B}}, \mathcal{E}_{A\to\bar{A}} \text{ quantum channels}$$

where we have the diamond norm distance

$$\|\mathcal{Q}_{A\to\bar{B}}\|_\diamond := \sup_{\|X\|_1\le 1} \|(\mathcal{Q}_{A\to\bar{B}} \otimes \mathcal{I}_A)(X_{AA})\|_1.$$

Employing the dual form of the Diamond norm distance [10, Sect. 4] together with the Choi isomorphism and its inverse

$$\left(J^{-1}(Q_{AB})\right)_{A\to B}(\cdot) := \mathrm{Tr}_A\left[Q_{AB}\left((\cdot)^T \otimes 1_B\right)\right]$$

this is rewritten as follows.

Lemma II.4. *Let $\mathcal{N}_{\bar{A}\to B}$ be a quantum channel and $M \in \mathbb{N}$. Then, we have that*

$$\begin{aligned}
&\Delta(\mathcal{N},\mathcal{I}_M)\\
&\quad = \min \lambda\\
&\qquad \text{s.t. } E_{A\bar{A}} \succeq 0,\ E_A = \frac{1_A}{d_A}\\
&\qquad\quad D_{B\bar{B}} \succeq 0,\ D_B = \frac{1_B}{d_B}\\
&\qquad\quad Z_{A\bar{B}} \succeq 0,\ \frac{\lambda}{d_A}\cdot 1_A \succeq Z_A\\
&\qquad\quad Z_{A\bar{B}} + \Phi_{A\bar{B}}\\
&\qquad\quad \succeq J\Big(J^{-1}\left(D_{B\bar{B}}\right) \circ \mathcal{N} \circ J^{-1}\left(E_{A\bar{A}}\right)\Big)_{A\bar{B}}.
\end{aligned}$$

We have $F(\mathcal{N}, M) \geq 1-\Delta(\mathcal{N},\mathcal{I}_M)$ and some equivalences follow from the considerations in [11]. For us the crucial point is that by inspection both for the average error (Lemma II.2) and maximum error setting (Lemma II.4) the optimization is over encoders and decoders with corresponding product Choi matrix $E_{A\bar{A}} \otimes D_{B\bar{B}}$.

III. De Finetti theorems for quantum channels

In order to give efficiently computable converse bounds for quantum coding our approach is to develop positive semidefinite representable outer approximations on the set of separable Choi states

$$\begin{aligned}
&\mathrm{SEP}_C(A\bar{A}|B\bar{B})\\
&:= \Bigg\{C_{A\bar{A}B\bar{B}} = \sum_{i\in I} p_i E^i_{A\bar{A}} \otimes D^i_{B\bar{B}}\ \Bigg|\ p_i \geq 0,\ \sum_{i\in I} p_i = 1,\\
&\qquad E_{A\bar{A}} \succeq 0,\ E_A = \frac{1_A}{d_A},\ D_{B\bar{B}} \succeq 0,\ D_B = \frac{1_B}{d_B}\Bigg\}.
\end{aligned}$$

Now, to characterize the set of separable operators is a fundamental and hard problem in quantum information theory (see, e.g., [12]). Nevertheless, the set of separable quantum states can be approximated by the sum-of-squares hierarchies of Lasserre [13] and Parrilo [14], as realised in the semidefinite programming hierarchy of Doherty-Parrilo-Spedalieri [15]. The underlying idea of the DPS hierarchy is that separable states ρ_{AB} are n-extendible to $\rho_{AB_1^n}$ for any n, where $B_1 \equiv B$, $B_1^n = B_1 \cdots B_n$ such that for any permutation π we have

$$\rho_{AB_1^n} = \left(\mathcal{I}_A \otimes \mathcal{U}^\pi_{B_1^n}\right)(\rho_{AB_1^n})$$

with $\mathcal{U}^\pi_{B_1^n}$ the map that permutes the systems B_1^n according to $\pi \in \mathfrak{S}_n$ = symmetric group of n elements. Due to the monogamy of entanglement, however, general quantum states do not have this property and finite quantum de Finetti theorems exactly quantify the distance of n-extendible states to separable states [16], with convergence in the limit $n \to \infty$.

For our setting, however, we are interested more generally in characterizing operators that are separable in the cut $A\bar{A}|B\bar{B}$ but subject to the linear Choi constraints on $A\bar{A}$ and $B\bar{B}$, respectively. Extending the entropic proof techniques from [17] we find the following de Finetti theorem for the set of separable Choi states.

Theorem III.1. *Let $W_{A\bar{A}(B\bar{B})_1^n}$ be a quantum state with*

$$\begin{aligned}
W_{A\bar{A}(B\bar{B})_1^n} &= (\mathcal{I}_{A\bar{A}} \otimes \mathcal{U}^\pi_{(B\bar{B})_1^n})(W_{A\bar{A}(B\bar{B})_1^n})\ \forall \pi \in \mathfrak{S}_n\\
W_{(B\bar{B})_1^{n-1}B_n} &= W_{(B\bar{B})_1^{n-1}} \otimes \frac{1_{B_n}}{d_B}\\
W_{A(B\bar{B})_1^n} &= \frac{1_A}{d_A} \otimes W_{(B\bar{B})_1^n}
\end{aligned}$$

where $\pi \in \mathfrak{S}_n$ denotes the symmetric group of n elements. Then, we have for $B \equiv B_1$ that

$$\begin{aligned}
&\inf_{C\in SEP_C} \|W_{AB\bar{A}\bar{B}} - C_{AB\bar{A}\bar{B}}\|_1\\
&\leq \min\left\{18\sqrt{d_A d_B}, d_B^2(d_B+1)\right\} \cdot \sqrt{\frac{(2\ln 2)\log\left(d_A\right)}{n}}
\end{aligned}$$

Proof. See the full version [9, Thm. 3.4]. □

The representation $W_{A\bar{A}B\bar{B}}$ we obtain in this theorem is close to a mixture of products of Choi states of completely positive and trace-preserving maps. We note that applying standard de Finetti theorems for quantum states would only show that $W_{A\bar{A}B\bar{B}}$ is close to a mixture of products of quantum states — or in other words Choi states of completely positive maps that are in general not even trace-non-increasing. This is not sufficient for our purpose, and the constraints in Theorem III.1 are needed in our proofs to achieve this stronger statement. We refer to the full version for a discussion of examples [9, Ex. 3.7].

We mention that by relating the trace norm distance of Choi states to the diamond norm distance of the quantum channels [18, Lem. 7] one can alternatively state the bounds from Theorem III.1 directly in terms of quantum channels [9, Cor. 3.8]. In particular, this leads to finite versions of the asymptotic de Finetti theorem for quantum channels from [19].

One of the main steps in the proof of Theorem III.1 is the use of informationally complete measurements for which the loss in distinguishability, or distortion, can be bounded. We give such a measurement for which we bound the distortion in the additional presence of quantum side information.

Lemma III.2. *Consider a state two-design on B, i.e., a set of rank-one projectors $\{P_B^z\}_{z\in\{1,\dots,t\}}$ such that*

$$\frac{1}{t}\sum_{z=1}^{t} P_B^z \otimes P_B^z = \frac{2P_{BB}^{\mathrm{sym}}}{d_B(d_B+1)}$$

where P_{BB}^{sym} denotes the projector onto the symmetric subspace of $B\otimes B$. Let $\mathcal{M}_B$ be the measurement defined as

$$\mathcal{M}_B(\cdot) = \sum_{z=1}^{t} \frac{d_B}{t}\cdot \mathrm{Tr}\big[P_B^z(\cdot)\big]|z\rangle\langle z|_B$$

and ξ_{AB} be a Hermitian operator on AB. Then, we have that

$$\|(\mathcal{I}_A\otimes\mathcal{M}_B)(\xi_{AB})\|_1 \geq \frac{1}{d_B^2(d_B+1)}\|\xi_{AB}\|_1.$$

Proof. See the full version [9, Lem. 3.3]. □

IV. Semidefinite program relaxations

The outer approximations of $\mathrm{SEP}_C(A\bar{A}|B\bar{B})$ from Theorem III.1 directly give hierarchies of efficiently computable semidefinite programming relaxations on the channel fidelity $F(\mathcal{N},M)$ via Lemma II.2, as well as the channel distance $\Delta(\mathcal{N},\mathcal{I}_M)$ via Lemma II.4. In the following we focus exclusively on the channel fidelity $F(\mathcal{N},M)$ for which the n-th level of the hierarchy becomes

$$\begin{aligned}
&\mathrm{sdp}_n(\mathcal{N},M)\\
&:= d_{\bar{A}}d_B\cdot\max\ \mathrm{Tr}\left[\left(J(\mathcal{N})_{\bar{A}B_1}\otimes\Phi_{A\bar{B}_1}\right)W_{A\bar{A}B_1\bar{B}_1}\right]\\
&\qquad \text{s.t. } W_{A\bar{A}(B\bar{B})_1^n}\succeq 0,\ \mathrm{Tr}[W_{A\bar{A}(B\bar{B})_1^n}]=1\\
&\qquad\quad W_{A\bar{A}(B\bar{B})_1^n} = (\mathcal{I}_{A\bar{A}}\otimes\mathcal{U}^\pi_{(B\bar{B})_1^n})(W_{A\bar{A}(B\bar{B})_1^n})\\
&\qquad\quad W_{A(B\bar{B})_1^n} = \frac{1_A}{d_A}\otimes W_{(B\bar{B})_1^n}\\
&\qquad\quad W_{A\bar{A}(B\bar{B})_1^{n-1}B_n} = W_{A\bar{A}(B\bar{B})_1^{n-1}}\otimes\frac{1_{B_n}}{d_B}
\end{aligned}$$

where we slightly strengthened the last condition by including the A-systems compared to the minimal condition dictated by Theorem III.1. We then immediately have the following monotone, asymptotic convergence.

Theorem IV.1. *Let $\mathcal{N}$ be a quantum channel and $n,M\in\mathbb{N}$. Then, we have that*

$$\mathrm{sdp}_{n+1}(\mathcal{N},M)\leq\mathrm{sdp}_n(\mathcal{N},M)$$
$$F(\mathcal{N},M)=\lim_{n\to\infty}\mathrm{sdp}_n(\mathcal{N},M).$$

The worst case convergence guarantee is slow, as to ensure that the approximation error becomes small, we need at least the level $n=\mathrm{poly}(d_Ad_BM)$.

This slow convergence in the worst case is as expected from the quantum separability problem [12]. Nevertheless, the relaxations $\mathrm{sdp}_n(\mathcal{N},M)$ inherit natural dimension bounds.

Lemma IV.2. *Let $\mathcal{N}_{\bar{A}\to B}$ be a quantum channel and $n,M\geq 1$. Then, we have that*

$$0\leq\mathrm{sdp}_n(\mathcal{N},M)\leq\min\left\{1,\left(\frac{d_{\bar{A}}}{M}\right)^2\right\}.$$

Proof. The lower bound is trivial. By the monotonicity in n (Theorem IV.1) it is enough to restrict to $n=1$ for the upper bounds. For the upper bounds we use that for any sub-normalized bipartite quantum state ρ_{XY} we have that $d_X\cdot 1_X\otimes\rho_Y\succeq\rho_{XY}$ [20, Lem. B.6]. For the first upper bound we find using the last constraint of $\mathrm{sdp}_1(\mathcal{N},M)$ that

$$\frac{d_{\bar{B}}}{d_B}\cdot W_{A\bar{A}}\otimes 1_{B_1\bar{B}_1}\succeq W_{A\bar{A}B_1\bar{B}_1}.$$

This in turn gives for the objective function

$$\begin{aligned}
&\mathrm{sdp}_1(\mathcal{N},M)\\
&\leq d_{\bar{A}}d_B\cdot\mathrm{Tr}\left[\left(J^{\mathcal{N}}_{\bar{A}B_1}\otimes\Phi_{A\bar{B}_1}\right)\left(\frac{d_{\bar{B}}}{d_B}\cdot W_{A\bar{A}}\otimes 1_{B_1\bar{B}_1}\right)\right]\\
&= d_{\bar{A}}d_{\bar{B}}\cdot\mathrm{Tr}\left[\left(\frac{1_A}{d_A}\otimes\frac{1_{\bar{A}}}{d_{\bar{A}}}\right)W_{A\bar{A}}\right]=\mathrm{Tr}\left[W_{A\bar{A}}\right]=1.
\end{aligned}$$

For the second upper bound we find similarly using the second to last constraint of $\mathrm{sdp}_1(\mathcal{N},M)$ that

$$\frac{d_{\bar{A}}}{d_A}\cdot 1_{A\bar{A}}\otimes W_{B_1\bar{B}_1}\succeq W_{A\bar{A}B_1\bar{B}_1}$$

which then leads to

$$\begin{aligned}
&\mathrm{sdp}_1(\mathcal{N},M)\\
&\leq\frac{d_{\bar{A}}^2d_B}{d_A}\cdot\mathrm{Tr}\left[\left(J^{\mathcal{N}}_{\bar{A}B_1}\otimes\Phi_{A\bar{B}_1}\right)\left(1_{A\bar{A}}\otimes W_{B_1\bar{B}_1}\right)\right]\\
&=\frac{d_{\bar{A}}^2d_B}{d_A}\cdot\mathrm{Tr}\left[\left(J^{\mathcal{N}}_{B_1}\otimes\frac{1_{\bar{B}_1}}{d_{\bar{B}}}\right)W_{B_1\bar{B}_1}\right]\\
&=\frac{d_{\bar{A}}^2d_B}{d_Ad_{\bar{B}}}\cdot\mathrm{Tr}\left[J^{\mathcal{N}}_{B_1}D_{B_1}\right]=\frac{d_{\bar{A}}^2}{d_Ad_{\bar{B}}}.
\end{aligned}$$

□

We can also add positive partial transpose (PPT) constraints

$$W^{T_{A\bar{A}}}_{A\bar{A}(B\bar{B})_1^n}\succeq 0,\ W^{T_{B_1\bar{B}_1}}_{A\bar{A}(B\bar{B})_1^n}\succeq 0,\dots,\ W^{T_{(B\bar{B})_1^{n-1}}}_{A\bar{A}(B\bar{B})_1^n}\succeq 0$$

where the partial transpose of an operator M_{AB} is defined for a fixed product basis as $\langle ij|M_{AB}^{T_A}|kl\rangle:=\langle kj|M_{AB}|il\rangle$. We denote the resulting relaxations by $\mathrm{sdp}_{n,\mathrm{PPT}}$ and it is an interesting question to study if the PPT constraints can lead to a faster convergence speed — cf. the discussion in [21]. For the relaxations $\mathrm{sdp}_{n,\mathrm{PPT}}$ we can then give a sufficient condition for exact convergence at a finite level of the hierarchy.

Lemma IV.3. *Let $W_{A\bar{A}(B\bar{B})_1^n}=(\mathcal{I}_{A\bar{A}}\otimes\mathcal{U}^\pi_{(B\bar{B})_1^n})(W_{A\bar{A}(B\bar{B})_1^n})$ for all $\pi\in\mathfrak{S}_n$ and fixed $0\leq k\leq n$ such that $W^{T_{(B\bar{B})_{k+1}^n}}_{A\bar{A}(B\bar{B})_1^n}\succeq 0$. If we have that*

$$\begin{aligned}
&\mathrm{rank}\left(W_{A\bar{A}(B\bar{B})_1^n}\right)\\
&\leq\max\left\{\mathrm{rank}\left(W_{A\bar{A}(B\bar{B})_1^k}\right),\mathrm{rank}\left(W_{(B\bar{B})_{k+1}^n}\right)\right\}
\end{aligned}$$

then $W_{A\bar{A}B\bar{B}}$ is separable with respect to $A\bar{A}|B\bar{B}$.

Proof. See the full version [9, Lem. 5.13]. □

We note that if the above criterion is fulfilled, then this also allows us to extract the optimal quantum encoder and decoder.

V. Numerical results

A. Symmetry reduction

For $n = 1$ we recover the outer bound from [4, Sect. IV], up to their a priori stronger condition

$$W_{AB} = \frac{1_{AB}}{d_A d_B} \text{ instead of our } \mathrm{Tr}\left[W_{A\bar{A}B\bar{B}}\right] = 1.$$

However, as implicitly shown in [4, Thm. 3] these two conditions actually become equivalent because of the structure of the objective function. Operationally $\mathrm{sdp}_1(\mathcal{N}, M)$ corresponds to the non-signalling assisted channel fidelity, whereas $\mathrm{sdp}_{1,\mathrm{PPT}}(\mathcal{N}, M)$ adds the PPT-preserving constraint — as discussed in [4, Cor. 4]. Moreover, in the objective function the symmetry[1]

$$\int \left(\overline{U}_A \otimes U_{\bar{B}}\right)(\cdot)\left(\overline{U}_A \otimes U_{\bar{B}}\right)^\dagger \, \mathrm{d}U$$

is used to achieve a dimension reduction of M^2 as described in [4, Thm. 3]

$$\begin{aligned}
&\mathrm{sdp}_{1,\mathrm{PPT}}(\mathcal{N}, M)\\
&\quad = d_{\bar{A}} d_B \cdot \max \mathrm{Tr}\left[J(\mathcal{N})_{\bar{A}B} Y_{\bar{A}B}\right]\\
&\qquad\qquad \text{s.t. } \rho_{\bar{A}} \otimes \frac{1_B}{d_B} \succeq Y_{\bar{A}B} \succeq 0,\ \mathrm{Tr}[\rho_{\bar{A}}] = 1\\
&\qquad\qquad M^2 \cdot Y_B = \frac{1_B}{d_B}\\
&\qquad\qquad \rho_{\bar{A}} \otimes \frac{1_B}{d_B} \succeq M \cdot Y_{\bar{A}B}^{T_B} \succeq -\rho_{\bar{A}} \otimes \frac{1_B}{d_B}\,.
\end{aligned}$$

To symmetrize $\mathrm{sdp}_{2,\mathrm{PPT}}(\mathcal{N}, M)$ for achieving a dimension reduction of M^3, one needs to compute the commutant of the action given by (cf. the discussion in [22])

$$\int \overline{U}_A \otimes U_{\bar{B}_1} \otimes U_{\bar{B}_2}(\cdot)\left(\overline{U}_A \otimes U_{\bar{B}_1} \otimes U_{\bar{B}_2}\right)^\dagger \, \mathrm{d}U\,.$$

B. Low dimensional studies

We performed proof of concept numerics to test the low levels of our hierarchy for sending one qubit ($M = 2$). The experiments have been done in MATLAB using CVX [23] and MOSEK [24]. It turns out that for all standard qubit and qutrit channels the first level $\mathrm{sdp}_{1,\mathrm{PPT}}$ already captures the channel fidelity in practice. In fact, numerically for $d_{\bar{A}} = d_B = 2, 3$ the approximations for separable Choi states from Theorem III.1 already seems to be exact for $n = 1$. This raises the possibility of a Peres-Horodecki type criterion for low dimensional quantum channels.

One main use of $\mathrm{sdp}_{2,\mathrm{PPT}}$ is to actually certify the exact optimality of $\mathrm{sdp}_{1,\mathrm{PPT}}$ via the rank-loop conditions in Lemma IV.3. However, in order to facilitate the search for solutions having rank loops we need to look for low rank solutions $W_{A\bar{A}(B\bar{B})_1^n}$, whereas typical solvers give high rank solutions. Nevertheless, a possible strategy is to find some solution $W_{A\bar{A}(B\bar{B})_1^n}$ and then to employ a heuristic to minimize the rank while keeping the hierarchy constraints. The heuristic we found the most effective for our purposes was the log-det method described in [25]. As a non-trivial example we found for the qubit bit flip channel with bit-flip probability $p = 0.1$ the optimal unitary encoders and decoders with Kraus operator $U_E = -|1\rangle\langle 0| + |0\rangle\langle 1|$ and $U_D = |0\rangle\langle 0| - |1\rangle\langle 1|$, respectively.

C. Qubit depolarizing channel

The qubit depolarizing channel for $p \in [0, 4/3]$ is given as

$$\mathrm{Dep}_2(p) : \rho_{\bar{A}} \mapsto p \cdot \mathrm{Tr}[\rho_{\bar{A}}]\frac{1_B}{2} + (1-p)\cdot \rho_B\,.$$

Following [26] one can exploit the symmetries of the N-fold qubit depolarizing channel to arrive the linear program

$$\begin{aligned}
&\mathrm{sdp}_{1,\mathrm{PPT}}\left(\mathrm{Dep}_2^{\otimes N}(p), 2\right)\\
&\quad = \max \sum_{i=0}^{N} \binom{N}{i}\left(1 - \frac{3p}{4}\right)^i \left(\frac{3p}{4}\right)^{N-i} m_i\\
&\qquad \text{s.t. } 0 \le m_i \le 1 \; i \in \{0, \ldots, N\}\\
&\qquad\quad -\frac{1}{2} \le \sum_{i=0}^{N} x_{i,k} m_i \le \frac{1}{2} \; k \in \{0, \ldots, N\}\\
&\qquad\quad \sum_{i=0}^{N} \binom{N}{i} 3^{N-i} m_i = 2^{2N-2}
\end{aligned}$$

where $x_{i,k} = \sum_{r=\max\{0,i+k-N\}}^{\min\{i,k\}} \binom{k}{r}\binom{N-k}{i-r}(-1)^{i-r}(d-1)^{k-r}(d+1)^{N-k+r-i}$. The results are depicted in Figure 1. We note that for $N = 5$ it seems that in the region $p \in [1, 4/3]$ the first level of the hierarchy exactly matches the see-saw lower bounds from [5, Fig. 3.7]. However, again for $N = 5$ and small p there seems to be a considerable gap between the upper bound $\mathrm{sdp}_{1,\mathrm{PPT}}\left(\mathrm{Dep}_2^{\otimes N}(p), 2\right)$ and the performance of the analytic 5 qubit stabilizer code from [27], see the full version [9, Fig. 2].

D. Qubit Amplitude damping channel

The qubit amplitude damping channel with $\gamma \in [0, 1]$ is given as

$$\begin{aligned}
&\mathrm{Amp}_2(\gamma) : \rho_{\bar{A}} \mapsto E_B^0 \rho_B {E_B^0}^\dagger + E_B^1 \rho_B {E_B^1}^\dagger\\
&\text{for } E_B^0 = |0\rangle\langle 0|_B + \sqrt{1-\gamma}|1\rangle\langle 1|_B,\ E_B^1 = \sqrt{\gamma}|0\rangle\langle 1|_B.
\end{aligned}$$

We computed $\mathrm{sdp}_{1,\mathrm{PPT}}\left(\mathrm{Amp}_2^{\otimes N}(\gamma), 2\right)$ for $N = 1, 2, 3, 4$ and the bounds are shown in Figure 2, where they are also compared with the fidelity of the trivial coding scheme, and the 4 qubit code from [28]. Notice the overlap between the first level of the hierarchy and the trivial coding scheme. Comparing these results with the see-saw type lower bounds from [5, Fig. 3.12] that significantly improve on the trivial coding scheme, we find that there is still a considerable gap to our upper bounds.

[1]Here, $\overline{U}_A$ denotes the complex conjugate of U_A with respect to some standard basis.

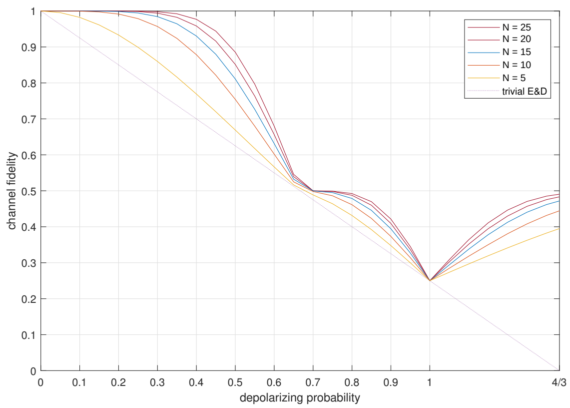

Fig. 1. Upper bounds $\mathrm{sdp}_{1,\mathrm{PPT}}\left(\mathrm{Dep}_2^{\otimes N}(p), 2\right)$ on the channel fidelity for $N = 5, 10, 15, 20, 25$ repetitions of the qubit depolarizing channel.

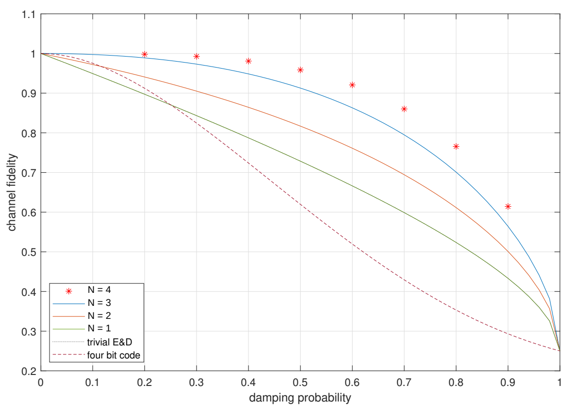

Fig. 2. Upper bounds $\mathrm{sdp}_{1,\mathrm{PPT}}\left(\mathrm{Amp}_2(\gamma)^{\otimes N}, 2\right)$ on the channel fidelity of the qubit amplitude damping channel for $N = 1, 2, 3, 4$ repetitions, the trivial encoder-decoder (same as $N = 1$), as well as the 4 qubit code from [28].

E. Outlook

We have provided some numerical evidence that the resulting bounds are tight for very low dimensional error models. More extensive numerical studies for practically relevant examples are the natural next step to explore. However, because of the rapidly growing dimensionality of the semidefinite program relaxations it becomes clear that numerical dimension reduction techniques exploiting symmetries are needed [29]. This is promising as the practically relevant examples are often highly symmetrical — such as, e.g., the N-fold qubit depolarizing channel.

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