

# De Finetti Theorems for Quantum Channels

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# Outline

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Motivation: Noisy Channel Coding

De Finetti Theorems

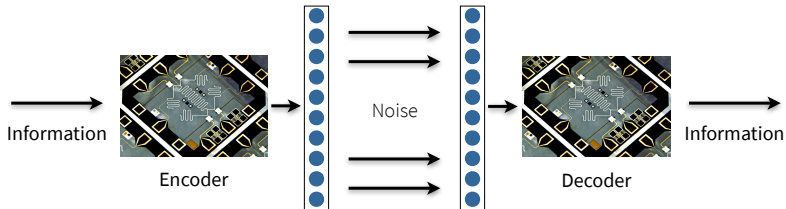
Application: Noisy Channel Coding

Conclusion

Proofs Ideas (board)

# Motivation: Noisy Channel Coding

# Noisy Channel Coding



## Error Correction

$m$  bits are subject to noise modelled by  $N(y|x)$ , find encoder  $e$  and decoder  $d$  to maximize probability  $p(N, m)$  of retrieving  $m$  bits

## Noisy Channel Coding (continued)

- ▶ Fixed number of bits  $m$  and noise model  $N$  gives **bilinear optimization**

$$\begin{aligned} p(N, m) &= \max_{(e, d)} \frac{1}{2^m} \sum_{x, y, i} N(y|x) d(i|y) e(x|i) \\ \text{s.t.} \quad &\sum_x e(x|i) = 1, \quad 0 \leq e(x|i) \leq 1 \\ &\sum_i d(i|y) = 1, \quad 0 \leq d(i|y) \leq 1 \end{aligned}$$

- ▶ Approximating  $p(N, m)$  up to multiplicative factor better than  $(1 - e^{-1})$  is **NP-hard** in the worst case [Barman & Fawzi 18]

## Noisy Channel Coding (continued)

- ▶ For the linear program [Hayashi 09, Polyanski *et al.* 10]

$$\begin{aligned} \text{lp}(N, m) &= \max_{(r,p)} \frac{1}{2^m} \sum_{x,y} N(y|x) r_{xy} \\ \text{s.t.} \quad & \sum_x r_{xy} \leq 1, \sum_x p_x = 2^m \\ & r_{xy} \leq p_x, 0 \leq r_{xy}, p_x \leq 1 \end{aligned}$$

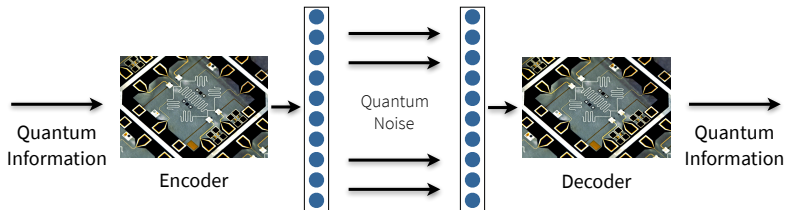
we have the approximation [Barman & Fawzi 18]

$$p(N, m) \leq \text{lp}(N, m) \leq \frac{1}{1 - e^{-1}} \cdot p(N, m)$$

- ▶ Outer bound complemented by **polynomial time**  $(1 - e^{-1})$ -multiplicative approximation algorithms

# Quantum Noisy Channel Coding

- ▶ Main question: Similar results for quantum error correction?  
[Matthews 12, Leung & Matthews 15]

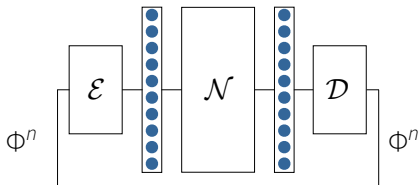


## Quantum Error Correction

Find encoder  $E$  and decoder  $D$  to maximize quantum probability  $F(\mathcal{N}, m)$  of retrieving  $m$  qubits

## Quantum Noisy Channel Coding (continued)

- ▶ Near-term quantum devices are of small and intermediate scale



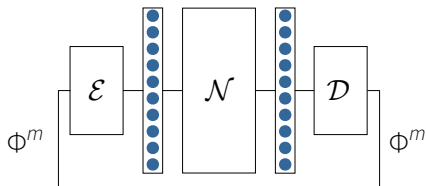
- ▶ Tailor-made **approximation algorithms** for encoder/ decoder?

Optimize Quantum Information Processing

Develop mathematical toolbox rooted in optimization theory



## Quantum Noisy Channel Coding (continued)



- ▶  $m$  qubits with quantum noise model  $\mathcal{N}$  leads to **quantum channel fidelity**

$$F(\mathcal{N}, m) := \max F\left(\Phi^m, ((\mathcal{D} \circ \mathcal{N} \circ \mathcal{E}) \otimes \mathcal{I})(\Phi^m)\right)$$

s.t.  $\mathcal{E}, \mathcal{D}$  quantum operations  
(+ physical constraints)

with fidelity  $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$ .

# Quantum Noisy Channel Coding (continued)

- ▶ Via Choi-Jamiołkowski becomes **bilinear optimization**

$$F(\mathcal{N}, m) = \max d_{\bar{A}} d_B \cdot \text{Tr} \left[ \left( J_{\bar{A}B}^{\mathcal{N}} \otimes \Phi_{\bar{A}\bar{B}} \right) (E_{\bar{A}\bar{A}} \otimes D_{\bar{B}\bar{B}}) \right]$$

s.t.  $E_{\bar{A}\bar{A}} \geq 0, E_A = \frac{1_A}{d_A}, D_{\bar{B}\bar{B}} \geq 0, D_B = \frac{1_B}{d_B}$

with product Choi states  $E_{\bar{A}\bar{A}} \otimes D_{\bar{B}\bar{B}}$  (where  $d_A = d_{\bar{B}} = 2^m$ ).

- ▶ At least as hard as classical problem, approximate solutions?
- ▶ **Inner bounds** on figure of merit via from any feasible solution via, e.g.,
  - ▶ physical intuition [Bennett *et al.* 96]
  - ▶ iterative see-saw methods [Reimpell & Werner 05]
- ▶ Main question: **Outer bounds**?

# Quantum Noisy Channel Coding (continued)

- ▶ Quantum channel fidelity stays the same with free shared randomness assistance

$$F(\mathcal{N}, n) = \max d_{\bar{A}} d_B \cdot \text{Tr} \left[ \left( J_{\bar{A}B}^{\mathcal{N}} \otimes \Phi_{\bar{A}\bar{B}} \right) \left( \sum_{i \in I} \rho_i E_{\bar{A}\bar{A}}^i \otimes D_{B\bar{B}}^i \right) \right]$$

s.t.  $\rho_i \geq 0, \sum_{i \in I} \rho_i = 1$

$$E_{\bar{A}\bar{A}}^i \geq 0, D_{B\bar{B}}^i \geq 0, E_A^i = \frac{1_A}{d_A}, D_B^i = \frac{1_B}{d_B} \quad \forall i \in I$$

and leads to separable Choi states  $\sum_{i \in I} \rho_i E_{\bar{A}\bar{A}}^i \otimes D_{B\bar{B}}^i$ .

- ▶ Idea: Find outer approximations on the set of **separable Choi states**  $\text{SEP}_{\mathcal{N}}(\bar{A}\bar{A}|B\bar{B})$
- ▶ NB: Strong hardness for quantum separability problem known [Barak *et al.* 12]

# De Finetti Theorems

# Monogamous Entanglement

- ▶ Quantum states  $\rho_{AB}$  are called  **$k$ -shareable** if

$$\rho_{AB_1^k} \equiv \rho_{AB_1 \dots B_k} \text{ with } \rho_{AB_j} = \rho_{AB} \quad \forall j \in [k]$$

⇒ characterizes separable states [Stoermer 69, Doherty *et al.* 02]

## De Finetti for Quantum States

For  $k$ -shareable quantum states  $\rho_{AB}$  we have that [Christandl *et al.* 07]

$$\left\| \rho_{AB} - \sum_{i \in I} p_i \sigma_A^i \otimes \omega_B^i \right\|_1 \leq \frac{d_B^2}{k}$$

with probabilities  $\{p_i\}_{i \in I}$  and quantum states  $\sigma_A^i, \omega_B^i \quad \forall i \in I$ .

- ▶ Use  $\rho_{AB_1^k} = (\mathcal{I}_A \otimes \pi_{B_1^k})(\rho_{AB_1^k})$  for  $\pi_{B_1^k} \in \mathfrak{S}_k$  — the symmetric group of  $k$  elements

# $k$ -shareable Choi States

## De Finetti for Choi States

For states  $\rho_{A\bar{A}(B\bar{B})_1^k} = (\mathcal{I}_{A\bar{A}} \otimes \pi_{(B\bar{B})_1^k})(\rho_{A\bar{A}(B\bar{B})_1^k}) \forall \pi_{(B\bar{B})_1^k} \in \mathfrak{S}_k$  with

$$\rho_{A(B\bar{B})_1^k} = \frac{1_A}{d_A} \otimes \rho_{(B\bar{B})_1^k} \quad \text{and} \quad \rho_{(B\bar{B})_1^{k-1}B_k} = \rho_{(B\bar{B})_1^{k-1}} \otimes \frac{1_{B_k}}{d_B}$$

we have that

$$\left\| \rho_{A\bar{A}B\bar{B}} - \sum_{i \in I} p_i \sigma_{A\bar{A}}^i \otimes \omega_{B\bar{B}}^i \right\|_1 \leq \sqrt{\frac{\text{poly}(d)}{k}} \quad \text{for } \sigma_A^i = \frac{1_A}{d_A}, \omega_B^i = \frac{1_B}{d_B} \quad \forall i \in I.$$

- ▶ Non-signalling:  $A \rightarrow (B\bar{B})_1^k$  and  $B_k \rightarrow (B\bar{B})_1^{k-1}$  [Duan & Winter 16]
- ▶ Set of  $k$ -shareable Choi states  $\text{SH}_{\mathcal{N}}^k(A\bar{A}|B\bar{B})$  characterizes set of separable Choi states  $\text{SEP}_{\mathcal{N}}(A\bar{A}|B\bar{B})$

# $k$ -shareable Quantum Channels

## De Finetti for Quantum Channels

For quantum channels with

$$\begin{aligned}\mathcal{N}_{AB_1^k \rightarrow \bar{A}\bar{B}_1^k} \left( \left( \mathcal{I}_A \otimes \pi_{B_1^k} \right) (\cdot) \right) &= \left( \mathcal{I}_A \otimes \pi_{\bar{B}_1^k} \right) \left( \mathcal{N}_{AB_1^k \rightarrow \bar{A}\bar{B}_1^k} (\cdot) \right) \quad \forall \pi_{B_1^k}, \pi_{\bar{B}_1^k} \in \mathfrak{S}_k \\ \mathrm{Tr}_{\bar{A}} \left[ \mathcal{N}_{AB_1^k \rightarrow \bar{A}\bar{B}_1^k} (\cdot) \right] &= \mathrm{Tr}_{\bar{A}} \left[ \mathcal{N}_{AB_1^k \rightarrow \bar{A}\bar{B}_1^k} \left( \frac{1_A}{d_A} \otimes \mathrm{Tr}_A[\cdot] \right) \right] \\ \mathrm{Tr}_{\bar{B}_k} \left[ \mathcal{N}_{AB_1^k \rightarrow \bar{A}\bar{B}_1^k} (\cdot) \right] &= \mathrm{Tr}_{\bar{B}_k} \left[ \mathcal{N}_{AB_1^k \rightarrow \bar{A}\bar{B}_1^k} \left( \mathrm{Tr}_{B_k}[\cdot] \otimes \frac{1_{B_k}}{d_B} \right) \right]\end{aligned}$$

we have that

$$\left\| \mathcal{N}_{AB \rightarrow \bar{A}\bar{B}} - \sum_{i \in I} p_i \mathcal{E}_{A \rightarrow \bar{A}}^i \otimes \mathcal{D}_{B \rightarrow \bar{B}}^i \right\|_{\diamond} \leq \sqrt{\frac{\mathrm{poly}(d)}{k}}$$

with probabilities  $\{p_i\}_{i \in I}$  and  $\mathcal{E}_{A \rightarrow \bar{A}}^i, \mathcal{D}_{B \rightarrow \bar{B}}^i$  quantum channels  $\forall i \in I$ .

# Application: Noisy Channel Coding



# Quantum Channel Fidelity

- ▶ Shared randomness assisted version

$$F(\mathcal{N}, m) = \max_{d_{\bar{A}} d_B} \cdot \text{Tr} \left[ \left( J_{\bar{A}B}^{\mathcal{N}} \otimes \Phi_{\bar{A}B} \right) \left( \sum_{i \in I} p_i E_{\bar{A}\bar{A}}^i \otimes D_{B\bar{B}}^i \right) \right]$$

$$\text{s.t. } p_i \geq 0, \sum_{i \in I} p_i = 1$$

$$E_{\bar{A}\bar{A}}^i \geq 0, D_{B\bar{B}}^i \geq 0, E_A^i = \frac{1_A}{d_A}, D_B^i = \frac{1_B}{d_B} \quad \forall i \in I$$

- ▶ Idea: Approximate separable Choi states  $\sum_{i \in I} p_i E_{\bar{A}\bar{A}}^i \otimes D_{B\bar{B}}^i$  by  $k$ -shareable Choi states  $W_{\bar{A}\bar{A}(B\bar{B})_1^k}$
- ▶  $k$ -shareable has **semi-definite representation**

# Outer Bound Approximations

- ▶ **Efficiently computable** semi-definite program outer bounds

$$\begin{aligned} \text{sdp}_k(\mathcal{N}, m) := \max \quad & d_{\bar{A}} d_B \cdot \text{Tr} \left[ (\mathcal{N}_{\bar{A} \rightarrow B_1}(\Phi_{\bar{A}\bar{A}}) \otimes \Phi_{\bar{A}B_1}) W_{\bar{A}\bar{A}B_1\bar{B}_1} \right] \\ \text{s.t.} \quad & W_{\bar{A}\bar{A}(B\bar{B})_1^k} \geq 0, \text{Tr} \left[ W_{\bar{A}\bar{A}(B\bar{B})_1^k} \right] = 1, W_{\bar{A}\bar{A}(B\bar{B})_1^k} = \left( \mathcal{I}_{\bar{A}\bar{A}} \otimes \pi_{(B\bar{B})_1^k} \right) \left( W_{\bar{A}\bar{A}(B\bar{B})_1^k} \right) \\ & W_{A(B\bar{B})_1^k} = \frac{1_A}{2^m} \otimes W_{(B\bar{B})_1^k}, W_{\bar{A}\bar{A}(B\bar{B})_1^{k-1}B_k} = W_{\bar{A}\bar{A}(B\bar{B})_1^{k-1}} \otimes \frac{1_{B_k}}{d_B} \\ & \text{PPT} \left( \bar{A}\bar{A} : (B\bar{B})_1^k \right) \geq 0 \quad (\text{positive partial transpose}) \end{aligned}$$

- ▶ With **approximation guarantee** to quantum channel fidelity

$$|\text{spd}_k(\mathcal{N}, m) - F(\mathcal{N}, m)| \leq \sqrt{\frac{\text{poly}(d)}{k}}$$

- ▶ Previous: [Matthews 12, Leung & Matthews 15, Tomamichel et al. 16, Wang et al. 16/17] and [Rozpedek et al. 18, Kaur et al. 18]

# Certifying Optimality of Relaxations

- ▶ Compare classical linear program relaxation [Barman & Fawzi 18]

$$p(N, m) \leq \text{lp}(N, m) \leq \frac{1}{1 - e^{-1}} \cdot p(N, m)$$

- ▶ No finite approximation guarantee for  $F(\mathcal{N}, m) \leq \text{sdp}_k(\mathcal{N}, m)$

## Rank Loop Conditions

If for  $k \in \mathbb{N}$  there exists  $l \in \mathbb{N}$  such that

$$\text{rank} \left( W_{A\bar{A}(B\bar{B})_1^k} \right) \leq \max \left\{ \text{rank} \left( W_{A\bar{A}(B\bar{B})_1^l} \right), \text{rank} \left( W_{(B\bar{B})_1^{k-l}} \right) \right\}$$

then we have equality  $\text{sdp}_k(\mathcal{N}, m) = F(\mathcal{N}, m)$

- ▶ Proof via [Navascués *et al.* 09]

# Numerical Example Relaxations

- ▶ Uniform noise corresponds to **qubit depolarizing channel**

$$\text{Dep}_p : \rho \mapsto p \cdot \frac{1_B}{2} + (1-p) \cdot \rho \quad \text{with } p \in [0, 4/3].$$

## Question

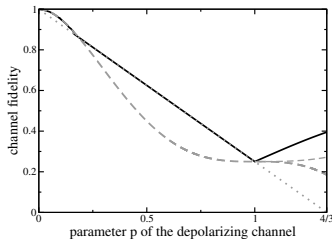
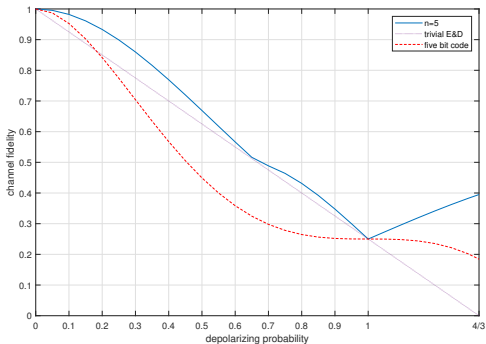
What is the optimal code for reliably storing  $m = 1$  qubit in noisy 5 qubit quantum memory, that is,  $\rho(\text{Dep}_p^{\otimes 5}, 1) = ?$

- ▶ Analytical [Bennett *et al.* 96] as well as see-saw [Reimpell & Werner 05] lower bounds, our work upper bounds

$$\rho(\text{Dep}_p^{\otimes 5}, 1) \leq \text{sdp}_k(\text{Dep}_p^{\otimes 5}, 1)$$

## Numerical Example Relaxations (continued)

- ▶ Exploiting symmetries for analytical **dimension reduction** for first level  $\text{sdp}_1(\text{Dep}_\rho^{\otimes 5}, 1)$  [Wang *et al.* 16/17] gives



[Reimpell & Werner 05] lower bounds

- ▶  $p \in [1, 4/3]$  [Reimpell & Werner 05] optimal,  $p \in [0, 0.18]$  room for improved codes

# Conclusion

# Conclusion

- ▶ Quantum noisy channel coding (one-shot) via **de Finetti theorem for quantum channels**
- ▶ Optimization theory tools to numerically study quantum error correction for practical settings
- ▶ Variations possible, e.g., classical communication assistance, physical constraints

## Open Questions

- ▶ Numerics via dimension reduction? Polynomial size + symdpoly numerics [Rosset]?
- ▶ Settings with **provably good quantum meta-converse**?
- ▶ Optimal quantum de Finetti: dimension dependence, minimal conditions?

## Proofs Ideas (board)



## Proof Ideas: De Finetti for Choi States

Let  $\rho_{A\bar{A}(B\bar{B})_1^k}$  be quantum states such that for  $\pi \in \mathfrak{S}_k$

$$\rho_{A\bar{A}(B\bar{B})_1^k} = (\mathcal{I}_{A\bar{A}} \otimes \pi_{(B\bar{B})_1^k})(\rho_{A\bar{A}(B\bar{B})_1^k})$$

$$\rho_{A(B\bar{B})_1^k} = \frac{1_A}{d_A} \otimes \rho_{(B\bar{B})_1^k}$$

$$\rho_{(B\bar{B})_1^{k-1}B_k} = \rho_{(B\bar{B})_1^{k-1}} \otimes \frac{1_{B_k}}{d_B}.$$

Then, we have that

$$\left\| \rho_{A\bar{A}B\bar{B}} - \sum_{i \in I} p_i \sigma_{A\bar{A}}^i \otimes \omega_{B\bar{B}}^i \right\|_1 \leq \sqrt{\frac{\text{poly}(d)}{k}}$$

with probabilities  $\{p_i\}_{i \in I}$  and  $\sigma_A^i = \frac{1_A}{d_A}$ ,  $\omega_B^i = \frac{1_B}{d_B} \forall i \in I$ .

## Proof Ideas: De Finetti with Linear Constraints

Let  $\rho_{AB_1^k}$  be quantum states,  $\Lambda_{A \rightarrow C_A}, \Gamma_{B \rightarrow C_B}$  linear maps, and  $X_{C_A}, Y_{C_B}$  operators such that for  $\pi \in \mathfrak{S}_k$

$$\begin{aligned} (\mathcal{I}_A \otimes \pi_{B_1^k}) (\rho_{AB_1^k}) &= \rho_{AB_1^k} && k\text{-shareable on } B \\ (\Lambda_{A \rightarrow C_A} \otimes \mathcal{I}_{B_1^k}) (\rho_{AB_1^k}) &= X_{C_A} \otimes \rho_{B_1^k} && \text{linear constraint on } A \\ \Gamma_{B_k \rightarrow C_B} (\rho_{B_1^k}) &= \rho_{B_1^{k-1}} \otimes Y_{C_B} && \text{linear constraint on } B. \end{aligned}$$

Then, we have that

$$\left\| \rho_{AB} - \sum_{i \in I} p_i \sigma_A^i \otimes \omega_B^i \right\|_1 \leq \sqrt{\frac{d_B^4 (d_B + 1)^2 \log d_A}{k}}$$

with probabilities  $\{p_i\}_{i \in I}$  and quantum states  $\sigma_A^i, \omega_B^i$  such that  $\forall i \in I$

$$\Lambda_{A \rightarrow C_A} (\sigma_A^i) = X_{C_A} \quad \text{and} \quad \Gamma_{B \rightarrow C_B} (\omega_B^i) = Y_{C_B}.$$

## Application: Bilinear Optimization

De Finetti with linear constraints gives **outer hierarchy for programs of the bilinear form**

$$\begin{aligned} \max \quad & \text{Tr}[H(D \otimes E)] \\ \text{s.t.} \quad & D \in \mathcal{S}_D, E \in \mathcal{S}_E \end{aligned}$$

where  $H$  is a matrix and  $\mathcal{S}_D$  and  $\mathcal{S}_E$  are positive semi-definite representable sets of the form

$$\mathcal{S}_D = \Pi_{A \rightarrow D}(\mathcal{S}_A^+ \cap \mathcal{A}_A) \quad \text{and} \quad \mathcal{S}_E = \Pi_{B \rightarrow E}(\mathcal{S}_B^+ \cap \mathcal{A}_B)$$

with  $\Pi_{A \rightarrow D}, \Pi_{B \rightarrow E}$  linear maps,  $\mathcal{S}_A^+, \mathcal{S}_B^+$  the set of density operators, and  $\mathcal{A}_A, \mathcal{A}_B$  affine subspaces of matrices.