Quantum Coding via Semidefinite Programming

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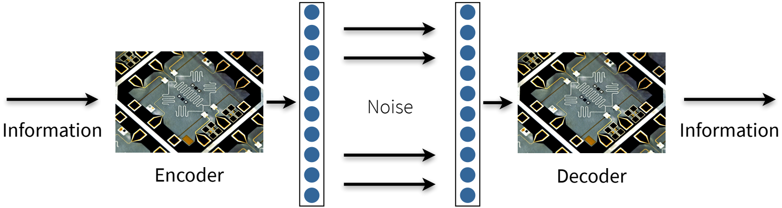
Noisy Channel Coding



Error Correction

m bits are subject to noise modelled by $N(y|x)$, find encoder e and decoder d to maximize probability $p(N, m)$ of retrieving m bits

Noisy Channel Coding (continued)

- Fixed number of bits m and noise model N gives **bilinear optimization**

$$
\begin{aligned}
p(N,m) = \max_{(e,d)} \quad & \frac{1}{2^m} \sum_{x,y,i} N(y|x)\,d(i|y)\,e(x|i) \\
\text{s.t.} \quad & \sum_x e(x|i) = 1,\ 0 \le e(x|i) \le 1 \\
& \sum_i d(i|y) = 1,\ 0 \le d(i|y) \le 1
\end{aligned}
$$

- Approximating $p(N,m)$ up to multiplicative factor better than $(1 - e^{-1})$ is **NP-hard** in the worst case [Barman & Fawzi 18].

Noisy Channel Coding (continued)

- For the linear program [Hayashi 09, Polyanski *et al.* 10]

$$\begin{aligned} \mathrm{lp}(N,m) = \max_{(r,p)} \quad & \frac{1}{2^m} \sum_{x,y} N(y|x) r_{xy} \\ \text{s.t.} \quad & \sum_x r_{xy} \le 1, \ \sum_x p_x = k \\ & r_{xy} \le p_x, \ 0 \le r_{xy}, p_x \le 1 \end{aligned}$$

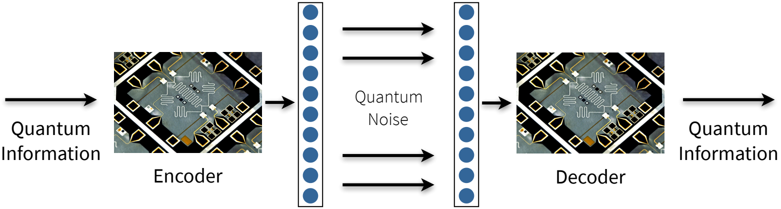
  we have the approximation [Barman & Fawzi 18]

$$p(N,m) \le \mathrm{lp}(N,m) \le \frac{1}{1-e^{-1}} \cdot p(N,m)$$

- **Polynomial-time** $(1 - e^{-1})$ additive approximation algorithms.
- Precise asymptotic bounds on the capacity of iid channels, etc.

Quantum Noisy Channel Coding

- Main question: Similar results for quantum error correction? [Matthews 12, Leung & Matthews 15]

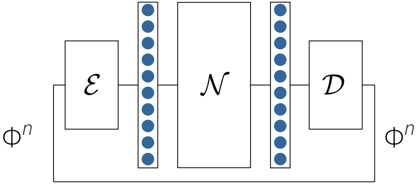


Quantum Error Correction

Find encoder E and decoder D to maximize probability $p(\mathcal{N}, m)$ of retrieving m qubits

Quantum Noisy Channel Coding (continued)

- Near-term quantum devices are of intermediate scale and noisy

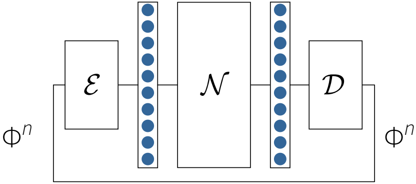


- Tailor-made **approximation algorithms** for encoder and decoder needed

Optimize Quantum Information Processing

Comprehensive practical mathematical toolbox rooted in optimization theory

Quantum Noisy Channel Coding (continued)



- Fixed number of qubits m and quantum noise model $\mathcal{N}$ leads to **quantum channel fidelity**

$$
\begin{aligned}
F(\mathcal{N}, n) := \max \quad & F\Big(\Phi^n, \big(\,(\mathcal{D} \circ \mathcal{N} \circ \mathcal{E}) \otimes \mathcal{I}\big)(\Phi^n)\Big) \\
\text{s.t.} \quad & \mathcal{E}, \mathcal{D} \text{ quantum operations} \\
& \text{(+ physical constraints)}
\end{aligned}
$$

Quantum Noisy Channel Coding (continued)

- For $d := \dim(\mathcal{N})$ becomes **bilinear optimization**

$$F(\mathcal{N}, n) = \max \quad d \cdot \mathrm{Tr}\Big[\Big(\mathcal{N}_{\bar{A}\to B}(\Phi_{\bar{A}\bar{A}}) \otimes \Phi_{A\bar{B}}\Big)\Big(\sum_{i\in I} p_i \mathcal{E}^i_{A\to\bar{A}} \otimes \mathcal{D}^i_{B\to\bar{B}}\Big)(\Phi_{AA} \otimes \Phi_{BB})\Big]$$

$$\text{s.t.} \quad \mathcal{E}^i, \mathcal{D}^i \text{ quantum operations}, \; p_i \geq 0, \; \sum_{i\in I} p_i = 1$$

- To characterize is set $\mathrm{SEP}_{\mathcal{N}}(A\bar{A}|B\bar{B})$ of **separable channels**

$$\sum_{i\in I} p_i \mathcal{E}^i_{A\to\bar{A}} \otimes \mathcal{D}^i_{B\to\bar{B}}$$

$\Rightarrow$ **strong hardness** for quantum separability problem [Barak *et al.* 12]

- Lower bounds on figure of merit via, e.g., physical intuition or iterative see-saw methods $\Rightarrow$ upper bounds?

Monogamous Entanglement

- De Finetti theorem for quantum states: ρ_{AB} k-**shareable** if

$$\rho_{AB_1 \dots B_k} \text{ with } \rho_{AB_j} = \rho_{AB} \ \forall j \in [k]$$

$\Rightarrow$ characterizes separable states [Stoermer 69] [Doherty *et al.* 02]

De Finetti Theorem for Quantum Channels

The set of separable channels is approximated by the set of k-shareable channels as

$$\left|\mathrm{SEP}_{\mathcal{N}}(A\bar{A}|B\bar{B}) - \mathrm{SH}^k_{\mathcal{N}}(A\bar{A}|B\bar{B})\right| \leq \sqrt{\frac{\mathcal{O}\left(d^3\right)}{k}}$$

where the set of k-shareable channels $\mathrm{SH}^k_{\mathcal{N}}$ has a semi-definite representation (cf. [Fuchs *et al.* 04, Kaur *et al.* 18]).

Monogamous Entanglement (continued)

- Non-commutative **sum-of-squares hierarchy** [Lasserre 00, Parrilo 03] via information-theoretic approach based on entropy inequalities [Brandão & Harrow 16]
- Efficiently computable **semi-definite program** outer bounds

$$\begin{aligned} \mathrm{sdp}_k(\mathcal{N}, m) := \max \quad & d_{\bar{A}} d_B \cdot \mathrm{Tr}\left[\left(\mathcal{N}_{\bar{A}\to B_1}\left(\Phi_{\bar{A}\bar{A}}\right) \otimes \Phi_{A\bar{B}_1}\right) W_{A\bar{A}B_1\bar{B}_1}\right] \\ s.t. \quad & W_{A\bar{A}(B\bar{B})_1^k} \geq 0,\ \mathrm{Tr}\left[W_{A\bar{A}(B\bar{B})_1^k}\right] = 1,\ \mathrm{PPT}\left(A_1^k : B_1^k\right) \geq 0 \\ & W_{A\bar{A}(B\bar{B})_1^k} = \left(\mathcal{I}_{A\bar{A}} \otimes \mathcal{U}^{\pi}_{(B\bar{B})_1^k}\right)\left(W_{A\bar{A}(B\bar{B})_1^k}\right) \quad \forall \pi \in \mathfrak{S}_k \\ & W_{A(B\bar{B})_1^k} = \frac{1_A}{2^m} \otimes W_{(B\bar{B})_1^k},\ W_{A\bar{A}(B\bar{B})_1^{k-1}B_k} = W_{A\bar{A}(B\bar{B})_1^{k-1}} \otimes \frac{1_{B_k}}{d_B} \end{aligned}$$

with **approximation guarantee** to quantum channel fidelity

$$\mathrm{spd}_k(\mathcal{N}, n) - F(\mathcal{N}, n) \leq \sqrt{\frac{\mathcal{O}\left(d_{\bar{A}}^2 d_B^8 \cdot \log d_A\right)}{k}}$$

Certifying Optimality of Relaxations

- Compare classical linear program relaxation [Barman & Fawzi 18]

$$p(N,m) \leq \mathrm{lp}(N,m) \leq \frac{1}{1-e^{-1}} \cdot p(N,m)$$

- No finite approximation guarantee for $F(\mathcal{N},m) \leq \mathrm{sdp}_k(\mathcal{N},m)$

Rank Loop Conditions

If for $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that

$$\mathrm{rank}\left(W_{A\bar{A}(B\bar{B})_1^k}\right) \leq \max\left\{\mathrm{rank}\left(W_{A\bar{A}(B\bar{B})_1^l}\right), \mathrm{rank}\left(W_{(B\bar{B})_1^{k-l}}\right)\right\}$$

then we have equality $\mathrm{sdp}_k(\mathcal{N},m) = F(\mathcal{N},m)$

- Proof via [Navascués *et al.* 09]

Numerical Example Relaxations

- Uniform noise corresponds to **qubit depolarizing channel**

$$\mathrm{Dep}_p : \rho \mapsto p \cdot \frac{1_B}{2} + (1-p) \cdot \rho \quad \text{with } p \in [0, 4/3].$$

Question

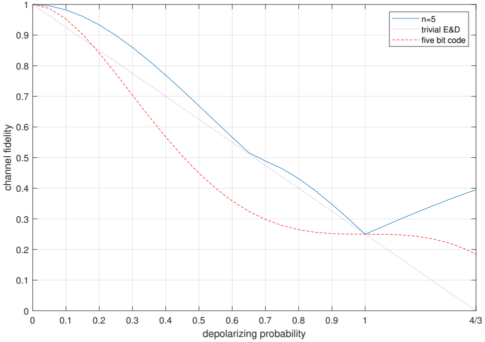
What is the optimal code for reliably storing $m = 1$ qubit in noisy 5 qubit quantum memory, $p\left(\mathrm{Dep}_p^{\otimes 5}, 1\right) = ?$

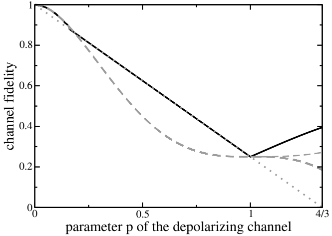
- Analytical [Bennett *et al.* 96] as well as numerical see-saw type [Reimpell & Werner 05] lower bounds available, our work gives

$$\mathrm{sdp}_k\left(\mathrm{Dep}_p^{\otimes 5}, 1\right) \geq p\left(\mathrm{Dep}_p^{\otimes 5}, 1\right)$$

Numerical Example Relaxations (continued)

- Exploiting symmetries for analytical **dimension reduction** for first level $\mathrm{sdp}_1\left(\mathrm{Dep}_p^{\otimes 5}, 1\right)$





See-saw lower bounds [Reimpell & Werner 05]

- For $p \in [0, 4/3]$ [Reimpell & Werner 05] optimal, for $p \in [0, 0.18]$ there is room to look for improved codes.

Conclusion

Take Home Message

Our optimization theory based approach provides tools to numerically study optimal quantum error correction for practically relevant settings of interest.

Outlook:

- So far theory outline work deriving generals methods
  $\Rightarrow$ general dimension reduction for numerics?
- Practical architectures and error models, better lower bounds?
- Optimal quantum de Finetti theorems?
- Settings with provably efficient approximations?