Strong Converse Bound on the Two-Way Assisted Quantum Capacity

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***Abstract*—We show that the max-Rains information of a quantum channel is an efficiently computable, single-letter strong converse upper bound for transmitting quantum information over quantum channels when assisted by positive-partial-transpose (PPT) preserving channels between every use of the channel. This includes in particular the quantum capacity with local operations and classical communication (LOCC) assistance.**

For our proof we make use of the amortized entanglement of quantum channels, which is defined as the largest net amount of entanglement that can be generated if the sender and receiver are allowed to share an arbitrary state before using the channel. Our main technical result is that amortization does not enhance the entanglement of quantum channels when entanglement is quantified by the max-Rains relative entropy. We prove this statement by employing semi-definite programming (SDP) duality and SDP formulations for the max-Rains relative entropy and the channel's max-Rains information, found recently in [Wang et al., arXiv:1709.00200].

I. Introduction and overview

Among the various asymptotic capacities of a quantum channel $\mathcal{N}$, the two-way assisted quantum capacity $Q_{\leftrightarrow}(\mathcal{N})$ is particularly relevant for tasks such as distributed quantum computation [1]. In the setting corresponding to this capacity, the sender and receiver are allowed to perform arbitrary local operations and classical communication (LOCC) between every use of the channel, and the capacity is equal to the maximum rate, measured in qubits per channel use, at which qubits can be transmitted reliably from the sender to the receiver [1]. Due to the teleportation protocol [2], this rate is equal to the maximum rate at which shared entangled bits can be generated reliably between the sender and the receiver [1]. The two-way assisted quantum capacity of certain channels such as the quantum erasure channel has been known for some time [3], but in general, it remains an open question to characterize $Q_{\leftrightarrow}(\mathcal{N})$.

Here, we are interested in placing upper bounds on the two-way assisted quantum capacity, and one way of simplifying the mathematics behind this task is to relax the class of free operations that the sender and receiver are allowed to perform between each channel use. With this in mind, we follow the approach of [4], [5] and relax the set LOCC to a larger class of operations known as PPT-preserving, standing for channels that are positive partial transpose preserving. The resulting capacity is then known as the PPT-assisted quantum capacity $Q_{\mathrm{PPT}}(\mathcal{N})$. It is equal to the maximum rate at which qubits can be communicated reliably from a sender to a receiver, when they are allowed to use a PPT-preserving channel in between every use of the actual channel $\mathcal{N}$. Figure 1 provides a visualization of such a PPT-assisted quantum communication protocol. Due to the containment LOCC $\subset$ PPT, the inequality $Q_{\leftrightarrow}(\mathcal{N}) \leq Q_{\mathrm{PPT}}(\mathcal{N})$ holds for all channels $\mathcal{N}$.

The starting point of our approach is to consider an entanglement measure $E(A;B)_\rho$ which is evaluated for a bipartite state ρ_{AB}. Given such an entanglement measure, one can define the entanglement $E(\mathcal{N})$ of a channel $\mathcal{N}$ by taking an optimization over all pure, bipartite states ψ_{RA} that could be input to the channel:

$$E(\mathcal{N}) = \sup_{\psi_{RA}} E(R;B)_\omega \quad \text{with } \omega_{RB} = \mathcal{N}_{A\to B}(\psi_{RA}). \quad (1)$$

The channel's entanglement $E(\mathcal{N})$ characterizes the amount of entanglement that a sender and receiver can generate by using the channel if they do not share entanglement prior to its use. Alternatively, one can consider the amortized entanglement $E_A(\mathcal{N})$ of a channel $\mathcal{N}$ as the optimization [6]

$$E_A(\mathcal{N}) = \sup_{\rho_{A'AB'}} E(A';BB')_\tau - E(A'A;B')_\rho, \quad (2)$$

where $\tau_{A'BB'} = \mathcal{N}_{A\to B}(\rho_{A'AB'})$ and $\rho_{A'AB'}$ is a state. The amortized entanglement quantifies the net amount of entanglement that can be generated by using the channel $\mathcal{N}$, if the sender and receiver are allowed to begin with some initial entanglement in the form of the state $\rho_{A'AB'}$. The inequality $E(\mathcal{N}) \leq E_A(\mathcal{N})$ always holds for any entanglement measure E but it is nontrivial if the opposite inequality $E_A(\mathcal{N}) \leq E(\mathcal{N})$ holds. This is known to occur for certain entanglement measures or for certain channels with particular symmetries [6]–[8]. One of the main observations of [6], connected to earlier developments in [8]–[12], is that the amortized entanglement of a channel serves as an upper bound on the entanglement of the final state ω_{AB} generated by an LOCC- or PPT-assisted quantum communication protocol that uses the channel n times:

$$E(A;B)_\omega \leq nE_A(\mathcal{N}). \quad (3)$$

Clearly, if the inequality $E_A(\mathcal{N}) \leq E(\mathcal{N})$ holds, then we have $E_A(\mathcal{N}) = E(\mathcal{N})$ and the upper bound becomes much simpler.

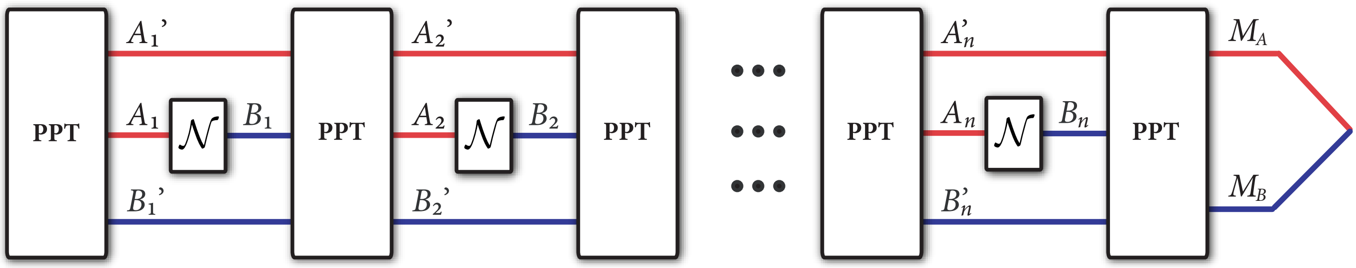

Fig. 1. A protocol for PPT-assisted quantum communication that uses a quantum channel n times. Every channel use is interleaved by a PPT-preserving channel. The goal of such a protocol is to produce an approximate maximally entangled state in the systems M_A and M_B, where the sender possesses system M_A and the receiver system M_B.

In this submission we show that the max-Rains information $R_{\max}(\mathcal{N})$ of a quantum channel $\mathcal{N}$ does not increase under amortization. That is, we prove that for all channels $\mathcal{N}$

$$R_{\max,A}(\mathcal{N}) = R_{\max}(\mathcal{N}), \tag{4}$$

where $R_{\max,A}(\mathcal{N})$ denotes the amortized max-Rains information. Note that $R_{\max}(\mathcal{N})$ and $R_{\max,A}(\mathcal{N})$ are respectively defined by taking the entanglement measure E in (1) and (2) to be the max-Rains relative entropy (which we define formally in the next section).

The max-Rains information was recently shown to be equal to an information quantity discussed in [13]–[15] based on semi-definite programming. To prove our main technical result – the equality in (4) – we critically make use of the tools and framework developed in these recent works. We note that the equality in (4) solves an open question posed in the conclusion of [8].

The main application of the equality in (4) is an efficiently computable, single-letter strong converse bound on $Q_{\text{PPT}}(\mathcal{N})$ as well as $Q_{\leftrightarrow}(\mathcal{N})$ for any channel $\mathcal{N}$. To arrive at this result, we simply apply the general inequality in (3) along with the equality in (4).

II. Concepts and Notation

Here, we provide some technical background needed for our derivations. For basic concepts and standard notation used in quantum information theory, we point the reader to [16].

Identities about quantum states and channels: Let $\mathcal{N}_{A\to B}$ be a quantum channel, and let $|\Upsilon\rangle_{RA}$ denote the maximally entangled vector

$$|\Upsilon\rangle_{RA} = \sum_i |i\rangle_R |i\rangle_A, \tag{5}$$

where the Hilbert spaces $\mathcal{H}_R$ and $\mathcal{H}_A$ are of the same dimension, and $\{|i\rangle_R\}_i$ and $\{|i\rangle_A\}_i$ are fixed orthonormal bases. The Choi operator for a channel $\mathcal{N}_{A\to B}$ is defined as

$$J^{\mathcal{N}}_{RB} = (\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(|\Upsilon\rangle\langle\Upsilon|_{RA}), \tag{6}$$

where id_R denotes the identity map on system R. One can recover the action of the channel $\mathcal{N}_{A\to B}$ on an arbitrary input state $\rho_{SA'}$ as

$$\langle\Upsilon|_{A'R}\ \rho_{SA'} \otimes J^{\mathcal{N}}_{RB}\ |\Upsilon\rangle_{A'R} = \mathcal{N}_{A\to B}(\rho_{SA}), \tag{7}$$

where A' is a system isomorphic to the channel input A. Another identity we recall is that

$$\langle\Upsilon|_{RA}\,(X_{SR}\otimes I_A)\,|\Upsilon\rangle_{RA} = \mathrm{Tr}_R\{X_{SR}\}, \tag{8}$$

for an operator X_{SR} acting on $\mathcal{H}_S \otimes \mathcal{H}_R$.

For a fixed basis $\{|i\rangle_B\}_i$ the partial transpose is the map

$$(\mathrm{id}_A \otimes T_B)(X_{AB}) = \sum_{i,j} (I_A \otimes |i\rangle\langle j|_B)\, X_{AB}\, (I_A \otimes |i\rangle\langle j|_B)\,,$$

where X_{AB} is an arbitrary operator acting on a tensor-product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. For simplicity we often employ the abbreviation $T_B(X_{AB}) = (\mathrm{id}_A \otimes T_B)(X_{AB})$. The partial transpose map plays a role in the following well known transpose trick identity:

$$(X_{SR}\otimes I_A)\,|\Upsilon\rangle_{RA} = (T_A(X_{SA})\otimes I_R)\,|\Upsilon\rangle_{RA}. \tag{9}$$

The partial transpose map plays another important role because a separable state

$$\sigma_{AB} = \sum_x p(x)\tau^x_A \otimes \omega^x_B \tag{10}$$

for a distribution $p(x)$ and states τ^x_A and ω^x_B, stays within the set of separable states $\mathrm{SEP}(A\!:\!B)$ under this map [17], [18]

$$T_B(\sigma_{AB}) \in \mathrm{SEP}(A\!:\!B). \tag{11}$$

This motivates defining the set of PPT states, which are those states σ_{AB} for which $T_B(\sigma_{AB}) \geq 0$. This in turn motivates defining the more general set of positive semi-definite operators [19]

$$\mathrm{PPT}'(A\!:\!B) = \{\sigma_{AB} : \sigma_{AB} \geq 0 \wedge \|T_B(\sigma_{AB})\|_1 \leq 1\}\,, \tag{12}$$

where we have employed the trace norm, defined for an operator X as $\|X\|_1 = \mathrm{Tr}\{|X|\}$ with $|X| = \sqrt{X^\dagger X}$. We then have the containments $\mathrm{SEP} \subset \mathrm{PPT} \subset \mathrm{PPT}'$.

An LOCC quantum channel $\mathcal{N}_{AB\to A'B'}$ consists of an arbitrarily large but finite number of compositions of the following:

1) Alice performs a quantum instrument, which has both a quantum and classical output. She forwards the classical

output to Bob, who then performs a quantum channel conditioned on the classical data received. This sequence of actions corresponds to a channel of the form

$$\sum_x \mathcal{F}^x_{A\to A'} \otimes \mathcal{G}^x_{B\to B'}, \tag{13}$$

where $\{\mathcal{F}^x_{A\to A'}\}_x$ is a collection of completely positive maps such that $\sum_x \mathcal{F}^x_{A\to A'}$ is a quantum channel and $\{\mathcal{G}^x_{B\to B'}\}_x$ is a collection of quantum channels.

2) The situation is reversed, with Bob performing the initial instrument, who forwards the classical data to Alice, who then performs a quantum channel conditioned on the classical data. This sequence of actions corresponds to a channel of the form in (13), with the A and B labels switched.

A quantum channel $\mathcal{N}_{AB\to A'B'}$ is a PPT-preserving channel if the map $T_{B'} \circ \mathcal{N}_{AB\to A'B'} \circ T_B$ is a quantum channel. Any LOCC channel is a PPT-preserving channel [4], [5].

Entropic measures: The max-relative entropy of a state ρ relative to a positive semi-definite operator σ is defined as [20]

$$D_{\max}(\rho\|\sigma) = \inf\left\{\lambda : \rho \leq 2^\lambda \sigma\right\}. \tag{14}$$

If $\operatorname{supp}(\rho) \not\subseteq \operatorname{supp}(\sigma)$, then $D_{\max}(\rho\|\sigma) = \infty$. The max-relative entropy is monotone non-increasing under the action of a quantum channel $\mathcal{N}$ in the sense that

$$D_{\max}(\rho\|\sigma) \geq D_{\max}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{15}$$

The max-Rains relative entropy of a state ρ_{AB} is defined as

$$R_{\max}(A;B)_\rho = \min_{\sigma_{AB}\in \mathrm{PPT}'(A:B)} D_{\max}(\rho_{AB}\|\sigma_{AB}), \tag{16}$$

and it is monotone non-increasing under the action of a PPT-preserving quantum channel $\mathcal{N}_{AB\to A'B'}$ in the sense that [21]

$$R_{\max}(A;B)_\rho \geq R_{\max}(A';B')_\omega, \tag{17}$$

for $\omega_{A'B'} = \mathcal{N}_{AB\to A'B'}(\rho_{AB})$. The max-Rains information of a quantum channel $\mathcal{N}_{A\to B}$ is defined by replacing E in (1) with the max-Rains relative entropy $R_{\max}$. That is,

$$R_{\max}(\mathcal{N}) = \max_{\phi_{SA}} R_{\max}(S;B)_\omega, \tag{18}$$

where $\omega_{SB} = \mathcal{N}_{A\to B}(\phi_{SA})$ and ϕ_{SA} is a pure state, with $|S| = |A|$. The amortized max-Rains information of a channel, denoted as $R_{\max,A}(\mathcal{N})$, is defined by replacing E in (2) with the max-Rains relative entropy $R_{\max}$.

The max-Rains relative entropy of a state ρ_{AB} can be expressed as [14, Eq. (8)] (see also [15])

$$R_{\max}(A;B)_\rho = \log_2 W(A;B)_\rho, \tag{19}$$

where $W(A;B)_\rho$ is the solution of the semi-definite program

$$\begin{aligned} \text{minimize} \quad & \operatorname{Tr}\{C_{AB} + D_{AB}\} \\ \text{subject to} \quad & C_{AB}, D_{AB} \geq 0 \\ & T_B(C_{AB} - D_{AB}) \geq \rho_{AB}. \end{aligned} \tag{20}$$

Similarly, in [15, Eq. (21)] the max-Rains information of a quantum channel $\mathcal{N}_{A\to B}$ is expressed as

$$R_{\max}(\mathcal{N}) = \log \Gamma(\mathcal{N}), \tag{21}$$

where $\Gamma(\mathcal{N})$ is the solution of the semi-definite program

$$\begin{aligned} \text{minimize} \quad & \|\operatorname{Tr}_B\{V_{SB} + Y_{SB}\}\|_\infty \\ \text{subject to} \quad & Y_{SB}, V_{SB} \geq 0 \\ & T_B(V_{SB} - Y_{SB}) \geq J^{\mathcal{N}}_{SB}. \end{aligned} \tag{22}$$

These formulations of $R_{\max}(A;B)_\rho$ and $R_{\max}(\mathcal{N})$ are the tools that we use to prove our main results. It is worthwhile to mention that they follow by employing the theory of semi-definite programming.

III. Main result

The following constitutes our main technical result.

Proposition 1: Let $\rho_{A'AB'}$ be a state and let $\mathcal{N}_{A\to B}$ be a quantum channel. Then, we have

$$R_{\max}(A';BB')_\omega \leq R_{\max}(\mathcal{N}) + R_{\max}(A'A;B')_\rho, \tag{23}$$

where $\omega_{A'BB'} = \mathcal{N}_{A\to B}(\rho_{A'AB'})$. Hence, amortization does not enhance the max-Rains information of a quantum channel:

$$R_{\max,A}(\mathcal{N}) = R_{\max}(\mathcal{N}). \tag{24}$$

Proof. By removing logarithms and applying (19) as well as (21), the desired inequality becomes equivalent to

$$W(A';BB')_\omega \leq \Gamma(\mathcal{N}) \cdot W(A'A;B')_\rho. \tag{25}$$

Exploiting the identity in (20), we find that

$$W(A'A;B')_\rho = \min \operatorname{Tr}\{C_{A'AB'} + D_{A'AB'}\}, \tag{26}$$

subject to the constraints

$$C_{A'AB'}, D_{A'AB'} \geq 0,\ T_{B'}(C_{A'AB'} - D_{A'AB'}) \geq \rho_{A'AB'}, \tag{27}$$

while the identity in (22) gives that

$$\Gamma(\mathcal{N}) = \min \|\operatorname{Tr}_B\{V_{SB} + Y_{SB}\}\|_\infty, \tag{28}$$

subject to the constraints

$$Y_{SB}, V_{SB} \geq 0,\ T_B(V_{SB} - Y_{SB}) \geq J^{\mathcal{N}}_{SB}. \tag{29}$$

The identity in (20) implies that the left-hand side of (25) is equal to

$$W(A';BB')_\omega = \min \operatorname{Tr}\{E_{A'BB'} + F_{A'BB'}\}, \tag{30}$$

subject to the constraints

$$E_{A'BB'}, F_{A'BB'} \geq 0 \tag{31}$$

$$T_{BB'}(E_{A'BB'} - F_{A'BB'}) \geq \mathcal{N}_{A\to B}(\rho_{A'AB'}). \tag{32}$$

With these SDP formulations in place, we can now establish the inequality in (25) by making judicious choices for $E_{A'BB'}$ and $F_{A'BB'}$. Let $C_{A'AB'}$ and $D_{A'AB'}$ be optimal for $W(A'A;B')_\rho$, and let Y_{SB} and V_{SB} be optimal for $\Gamma(\mathcal{N})$. Let

$|\Upsilon\rangle_{SA}$ be the maximally entangled vector, as defined in (5). Pick the operators

$$E_{A'BB'} = \langle\Upsilon|_{SA} C_{A'AB'} \otimes V_{SB} + D_{A'AB'} \otimes Y_{SB} |\Upsilon\rangle_{SA} \tag{33}$$

$$F_{A'BB'} = \langle\Upsilon|_{SA} C_{A'AB'} \otimes Y_{SB} + D_{A'AB'} \otimes V_{SB} |\Upsilon\rangle_{SA}. \tag{34}$$

We then have that $E_{A'BB'}, F_{A'BB'} \geq 0$ because $C_{A'AB'}$, $D_{A'AB'}$, Y_{SB}, $V_{SB} \geq 0$. Now, consider that

$$\begin{aligned} &T_{BB'}(E_{A'BB'} - F_{A'BB'}) \\ &= T_{BB'}\left[\langle\Upsilon|_{SA}(C_{A'AB'} - D_{A'AB'}) \otimes (V_{SB} - Y_{SB})|\Upsilon\rangle_{SA}\right] \\ &= \langle\Upsilon|_{SA} T_{B'}(C_{A'AB'} - D_{A'AB'}) \otimes T_B(V_{SB} - Y_{SB})|\Upsilon\rangle_{SA} \\ &\geq \langle\Upsilon|_{SA}\ \rho_{A'AB'} \otimes J^{\mathcal{N}}_{SB}|\Upsilon\rangle_{SA} \\ &= \mathcal{N}_{A\to B}(\rho_{A'AB'}), \end{aligned} \tag{35}$$

where the inequality follows from (27) and (29), and the last equality follows from (7). Also consider that

$$\begin{aligned} &\mathrm{Tr}\{E_{A'BB'} + F_{A'BB'}\} \\ &= \mathrm{Tr}\{\langle\Upsilon|_{SA}(C_{A'AB'} + D_{A'AB'}) \otimes (V_{SB} + Y_{SB})|\Upsilon\rangle_{SA}\} \\ &= \mathrm{Tr}\{(C_{A'AB'} + D_{A'AB'}) T_A(V_{AB} + Y_{AB})\} \\ &= \mathrm{Tr}\{(C_{A'AB'} + D_{A'AB'}) T_A(\mathrm{Tr}_B\{V_{AB} + Y_{AB}\})\} \\ &\leq \mathrm{Tr}\{C_{A'AB'} + D_{A'AB'}\} \left\|T_A(\mathrm{Tr}_B\{V_{AB} + Y_{AB}\})\right\|_\infty \\ &= \mathrm{Tr}\{C_{A'AB'} + D_{A'AB'}\} \left\|\mathrm{Tr}_B\{V_{AB} + Y_{AB}\}\right\|_\infty \\ &= W(A'A;B')_\rho \cdot \Gamma(\mathcal{N}). \end{aligned} \tag{36}$$

The second equality follows from (9) and (8). The inequality is a consequence of Hölder's inequality. The final equality follows because the spectrum of an operator is invariant under the action of a (full) transpose. Thus, we conclude that our choices of $E_{A'BB'}$ and $F_{A'BB'}$ are feasible for $W(A';BB')_\omega$. Since $W(A';BB')_\omega$ involves a minimization over all $E_{A'BB'}$ and $F_{A'BB'}$ satisfying (31) and (32), this concludes our proof of (25). ■

IV. PPT- and LOCC-assisted communication

We begin by reviewing the structure of a PPT-assisted quantum communication protocol (see Figure 1), along the lines discussed in [6]. In such a protocol, a sender Alice and a receiver Bob are spatially separated and connected by a quantum channel $\mathcal{N}_{A\to B}$. They begin by performing a PPT channel $\mathcal{P}^{(1)}_{\emptyset\to A'_1A_1B'_1}$, which leads to a PPT state $\rho^{(1)}_{A'_1A_1B'_1}$, where the system A_1 is such that it can be fed into the first channel use. Alice sends system A_1 through the first channel use, leading to a state $\sigma^{(1)}_{A'_1B_1B'_1} \equiv \mathcal{N}_{A_1\to B_1}\left(\rho^{(1)}_{A'_1A_1B'_1}\right)$. Alice and Bob then perform the PPT channel $\mathcal{P}^{(2)}_{A'_1B_1B'_1\to A'_2A_2B'_2}$, which leads to the state

$$\rho^{(2)}_{A'_2A_2B'_2} \equiv \mathcal{P}^{(2)}_{A'_1B_1B'_1\to A'_2A_2B'_2}\left(\sigma^{(1)}_{A'_1B_1B'_1}\right). \tag{37}$$

Alice sends system A_2 through the second channel use $\mathcal{N}_{A_2\to B_2}$, leading to the state $\sigma^{(2)}_{A'_2B_2B'_2} \equiv \mathcal{N}_{A_2\to B_2}\left(\rho^{(1)}_{A'_2A_2B'_2}\right)$. This process iterates: the protocol uses the channel n times. In general, we have for all $i \in \{2,\ldots,n\}$ the states

$$\rho^{(i)}_{A'_iA_iB'_i} \equiv \mathcal{P}^{(i)}_{A'_{i-1}B_{i-1}B'_{i-1}\to A'_iA_iB'_i}\left(\sigma^{(i-1)}_{A'_{i-1}B_{i-1}B'_{i-1}}\right) \tag{38}$$

$$\sigma^{(i)}_{A'_iB_iB'_i} \equiv \mathcal{N}_{A_i\to B_i}\left(\rho^{(i)}_{A'_iA_iB'_i}\right), \tag{39}$$

where $\mathcal{P}^{(i)}_{A'_{i-1}B_{i-1}B'_{i-1}\to A'_iA_iB'_i}$ is a PPT channel. The final step of the protocol consists of a PPT channel $\mathcal{P}^{(n+1)}_{A'_nB_nB'_n\to M_AM_B}$ which generates the systems M_A and M_B for Alice and Bob, respectively. The protocol's final state is

$$\omega_{M_AM_B} \equiv \mathcal{P}^{(n+1)}_{A'_nB_nB'_n\to M_AM_B}\left(\sigma^{(n)}_{A'_nB_nB'_n}\right). \tag{40}$$

The goal of the protocol is that the final state $\omega_{M_AM_B}$ is close to a maximally entangled state. Fix $n, M \in \mathbb{N}$ and $\varepsilon \in [0,1]$. The original protocol is an (n, M, ε) protocol if the channel is used n times as discussed above, $|M_A| = |M_B| = M$, and if

$$\begin{aligned} F(\omega_{M_AM_B}, \Phi_{M_AM_B}) &= \langle\Phi|_{M_AM_B}\omega_{M_AM_B}|\Phi\rangle_{M_AM_B} \\ &\geq 1-\varepsilon, \end{aligned} \tag{41}$$

where we have Uhlmann's fidelity $F(\tau,\kappa) \equiv \|\sqrt{\tau}\sqrt{\kappa}\|_1^2$ and the maximally entangled state $\Phi_{M_AM_B} = |\Phi\rangle\langle\Phi|_{M_AM_B}$ with

$$|\Phi\rangle_{M_AM_B} \equiv \frac{1}{\sqrt{M}}\sum_{m=1}^{M}|m\rangle_{M_A}\otimes|m\rangle_{M_B}. \tag{42}$$

A rate R is achievable for PPT-assisted quantum communication if for all $\varepsilon \in (0,1]$, $\delta > 0$, and sufficiently large n, there exists an $\left(n, 2^{n(R-\delta)}, \varepsilon\right)$ protocol. The PPT-assisted quantum capacity of a channel $\mathcal{N}$, denoted as $Q_{\mathrm{PPT}}(\mathcal{N})$, is equal to the supremum of all achievable rates. Moreover, a rate R is a strong converse rate for PPT-assisted quantum communication if for all $\varepsilon \in [0,1)$, $\delta > 0$, and sufficiently large n, there does not exist an $\left(n, 2^{n(R+\delta)}, \varepsilon\right)$ protocol. We can also consider the whole development above when we only allow the assistance of LOCC channels instead of PPT channels. In this case, we have similar notions as above, and then we arrive at the LOCC-assisted quantum capacity $Q_{\leftrightarrow}(\mathcal{N})$.

We now prove the following upper bound on the communication rate $Q = \frac{1}{n}\log_2 M$ (qubits per channel use) of any (n, M, ε) PPT-assisted protocol.

Theorem 2: Fix $n, M \in \mathbb{N}$ and $\varepsilon \in (0,1)$. Then, we have for any (n, M, ε) protocol for PPT-assisted quantum communication over a quantum channel $\mathcal{N}$ that

$$\log_2 M \leq nR_{\max}(\mathcal{N}) + \log_2\left(\frac{1}{1-\varepsilon}\right), \tag{43}$$

which is equivalent to $1-\varepsilon \leq 2^{-n[Q-R_{\max}(\mathcal{N})]}$.

Proof. From the assumption in (41) we have $\mathrm{Tr}\{\Phi_{M_AM_B}\omega_{M_AM_B}\} \geq 1-\varepsilon$, while [4, Lemma 2] implies that

$$\mathrm{Tr}\{\Phi_{M_AM_B}\sigma_{M_AM_B}\} \leq \frac{1}{M}\ \forall\sigma_{M_AM_B} \in \mathrm{PPT}'(M_A\!:\!M_B). \tag{44}$$

So under an "entanglement test"– a measurement of the form $\{\Phi_{M_A M_B}, I_{M_A M_B} - \Phi_{M_A M_B}\}$ – and applying the data processing inequality for the max-relative entropy, we get

$$R_{\max}(M_A; M_B)_\omega \geq \log_2 [(1 - \varepsilon) M]. \tag{45}$$

From the monotonicity of the Rains relative entropy with respect to PPT-preserving channels [5], [21], we find that

$$\begin{aligned}
&R_{\max}(M_A; M_B)_\omega \\
&\leq R_{\max}(A'_n; B_n B'_n)_{\sigma^{(n)}} \\
&= R_{\max}(A'_n; B_n B'_n)_{\sigma^{(n)}} - R_{\max}(A'_1 A_1; B'_1)_{\rho^{(1)}} \\
&= R_{\max}(A'_n; B_n B'_n)_{\sigma^{(n)}} \\
&\quad + \left[\sum_{i=2}^{n} R_{\max}(A'_i A_i; B'_i)_{\rho^{(i)}} - R_{\max}(A'_i A_i; B'_i)_{\rho^{(i)}}\right] \\
&\quad - R_{\max}(A'_1 A_1; B'_1)_{\rho^{(1)}} \\
&\leq \sum_{i=1}^{n} R_{\max}(A'_i; B_i B'_i)_{\sigma^{(i)}} - R_{\max}(A'_i A_i; B'_i)_{\rho^{(i)}} \\
&\leq n R_{\max}(\mathcal{N}).
\end{aligned} \tag{46}$$

The first equality follows because the state $\rho^{(1)}_{A'_1 A_1 B'_1}$ is a PPT state with vanishing max-Rains relative entropy. The second equality follows by adding and subtracting terms. The second inequality follows because $R_{\max}(A'_i A_i; B'_i)_{\rho^{(i)}} \leq R_{\max}(A'_{i-1}; B_{i-1} B'_{i-1})_{\sigma^{(i-1)}}$ for all $i \in \{2, \ldots, n\}$, due to monotonicity of the Rains relative entropy with respect to PPT channels. The final inequality follows by applying Prop. 1 to each term $R_{\max}(A'_n; B_n B'_n)_{\sigma^{(i)}} - R_{\max}(A'_i A_i; B'_i)_{\rho^{(i)}}$. Combining (45) and (46), we arrive at the inequality in (43). ■

Hence, the max-Rains information is a single-letter strong converse bound on the PPT-assisted quantum capacity. It is "single-letter" in the sense that the max-Rains information only requires an optimization over a single use of the channel.

V. Discussion

The main contribution of this submission was to show that the max-Rains information of a quantum channel does not increase under amortization. This result then implies an efficiently computable, single-letter strong converse bound on the capacity of a quantum channel to communicate qubits along with the assistance of PPT-preserving operations between every channel use. As such, the max-Rains information can be easily evaluated and is a general benchmark for the two-way assisted quantum capacity. The main tool that we used is the formulation of the max-Rains relative entropy and max-Rains information as semi-definite programs [13]–[15]. We discuss in [22] how our strong converse result stands with respect to prior work on strong converses of quantum and private capacities. There, we also provide an elementary proof for the fact that amortization does not enhance a channel's max-relative entropy of entanglement – leading to a strong converse upper bound on the two-way assisted private capacity $P_{\leftrightarrow}(\mathcal{N})$ of any quantum channel $\mathcal{N}$ (this latter result was first proven in [8] based on complex interpolation theory).

Acknowledgment

We are grateful to Omar Fawzi, Xin Wang, David Reeb, Siddhartha Das, and Andreas Winter for discussions related to the topic of this paper. MMW acknowledges support from the Office of Naval Research and the National Science Foundation.

References

[1] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, "Mixed-state entanglement and quantum error correction," *Physical Review A*, vol. 54, no. 5, pp. 3824–3851, 1996.

[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, "Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels," *Physical Review Letters*, vol. 70, no. 13, pp. 1895–1899, 1993.

[3] C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin, "Capacities of quantum erasure channels," *Physical Review Letters*, vol. 78, no. 16, pp. 3217–3220, 1997.

[4] E. M. Rains, "Bound on distillable entanglement," *Physical Review A*, vol. 60, no. 1, pp. 179–184, 1999.

[5] ——, "A semidefinite program for distillable entanglement," *IEEE Transactions on Information Theory*, vol. 47, no. 7, pp. 2921–2933, 2001.

[6] E. Kaur and M. M. Wilde, "Amortized entanglement of a quantum channel and approximately teleportation-simulable channels," *Journal of Physics A*, vol. 51, no. 3, p. 035303, 2018.

[7] M. Takeoka, S. Guha, and M. M. Wilde, "The squashed entanglement of a quantum channel," *IEEE Transactions on Information Theory*, vol. 60, no. 8, pp. 4987–4998, 2014.

[8] M. Christandl and A. Müller-Hermes, "Relative entropy bounds on quantum, private and repeater capacities," *Communications in Mathematical Physics*, vol. 353, no. 2, pp. 821–852, 2017.

[9] C. H. Bennett, A. W. Harrow, D. W. Leung, and J. A. Smolin, "On the capacities of bipartite Hamiltonians and unitary gates," *IEEE Transactions on Information Theory*, vol. 49, no. 8, pp. 1895–1911, 2003.

[10] M. S. Leifer, L. Henderson, and N. Linden, "Optimal entanglement generation from quantum operations," *Physical Review A*, vol. 67, no. 1, p. 012306, 2003.

[11] K. Ben Dana, M. García Díaz, M. Mejatty, and A. Winter, "Resource theory of coherence: Beyond states," *Physical Review A*, vol. 95, no. 6, p. 062327, 2017.

[12] L. Rigovacca, G. Kato, S. Baeuml, M. S. Kim, W. J. Munro, and K. Azuma, "Versatile relative entropy bounds for quantum networks," July 2017, arXiv:1707.05543.

[13] X. Wang and R. Duan, "A semidefinite programming upper bound of quantum capacity," *IEEE International Symposium on Information Theory (ISIT)*, pp. 1690–1694, 2016.

[14] ——, "An improved semidefinite programming upper bound on distillable entanglement," *Physical Review A*, vol. 94, no. 5, p. 050301, 2016.

[15] X. Wang, K. Fang, and R. Duan, "Semidefinite programming converse bounds for quantum communication," 2017, arXiv:1709.00200.

[16] M. M. Wilde, *From Classical to Quantum Shannon Theory*, 2016, arXiv:1106.1445v7.

[17] M. Horodecki, P. Horodecki, and R. Horodecki, "Separability of mixed states: necessary and sufficient conditions," *Physics Letters A*, vol. 223, no. 1-2, pp. 1–8, 1996.

[18] A. Peres, "Separability criterion for density matrices," *Physical Review Letters*, vol. 77, no. 8, pp. 1413–1415, 1996.

[19] K. Audenaert, B. De Moor, K. G. H. Vollbrecht, and R. F. Werner, "Asymptotic relative entropy of entanglement for orthogonally invariant states," *Physical Review A*, vol. 66, no. 3, p. 032310, 2002.

[20] N. Datta, "Min- and max-relative entropies and a new entanglement monotone," *IEEE Transactions on Information Theory*, vol. 55, no. 6, pp. 2816–2826, 2009.

[21] M. Tomamichel, M. M. Wilde, and A. Winter, "Strong converse rates for quantum communication," *IEEE Transactions on Information Theory*, vol. 63, no. 1, pp. 715–727, 2017.

[22] M. Berta and M. M. Wilde, "Amortization does not enhance the max-Rains information of a quantum channel," 2017, arXiv:1709.04907.