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Modern Topics in Quantum Information Theory
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Matrix Trace Inequalities for Quantum Entropy

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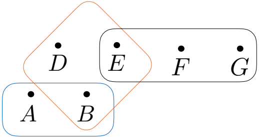
based on joint work with Brandão, Fawzi, Hirche, Sutter, Tomamichel

Motivation

- **Entropy** of quantum states ρ_A on Hilbert spaces $\mathcal{H}_A$ [von Neumann 27]:

$$H(A)_\rho := -\mathrm{tr}\left[\rho_A \log \rho_A\right]$$



Entropy for multipartite quantum systems: **entanglement leads to non-commutativity**

- **Mathematical properties** of quantum entropy from:

Strong subadditivity (SSA) [Lieb, Ruskai 73]

For tripartite quantum states ρ_{ABC} on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ we have

$$H(AB)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(B)_\rho$$

- SSA is an essential tool in all of quantum information science since it gives constraints on **multipartite entanglement structure**

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Motivation

- Understanding quantum entropy goes hand in hand with **matrix trace inequalities**, e.g., for H_1, H_2 Hermitian

$$\exp(H_1)\exp(H_2) = \exp(H_1 + H_2 + \frac{1}{2}[H_1, H_2] + \dots) \quad \text{Baker-Campbell-Hausdorff}$$

$$\operatorname{tr}[\exp(H_1)\exp(H_2)] \geq \operatorname{tr}[\exp(H_1 + H_2)] \quad \text{Golden-Thompson}$$

- Entropy for multipartite quantum systems requires **multivariate trace inequalities**:

SSA from Lieb's triple matrix inequality [Lieb 73]

For H_1, H_2, H_3 Hermitian we have

$$\int_0^\infty \operatorname{tr}\left[\exp(H_1)\frac{1}{\exp(-H_3)+\lambda}\exp(H_2)\frac{1}{\exp(-H_3)+\lambda}\right]\mathrm{d}\lambda \geq \operatorname{tr}[\exp(H_1 + H_2 + H_3)]$$

$$\left(\text{commutative case: } \int_0^\infty \frac{1}{(x^{-1}+\lambda)^2}\,\mathrm{d}\lambda = x\right)$$

Overview

1. From classical to quantum entropy
2. Entropy inequalities
3. Multivariate trace inequalities
4. Proofs via complex interpolation theory
5. Proof of entropy inequalities
6. Outlook

Entropy for classical systems

- **Entropy** of probability distribution P of random variable X over finite alphabet [Shannon 48, Rényi 61]:

$$H(X)_P := -\sum_x P(x)\log P(x) \quad \text{with } P(x)\log P(x) = 0 \text{ for } P(x) = 0$$

- **Relative entropy** of P with respect to distribution Q over finite alphabet,

$$D(P\|Q) := \sum_x P(x)\log\frac{P(x)}{Q(x)} \quad \text{[Kullback, Leibler 51]}$$

where $P(x)\log\frac{P(x)}{Q(x)} = 0$ for $P(x) = 0$ and by continuity $+\infty$ if $P \not\ll Q$.

- **Multipartite entropy measures** are generated through relative entropy, e.g., SSA:

$$H(XY)_P + H(YZ)_P \geq H(XYZ)_P + H(Y)_P \quad \text{equivalent to}$$
$$D(P_{XYZ}\|U_X \times P_{YZ}) \geq D(P_{XY}\|U_X \times P_Y) \quad \text{with } U_X \text{ uniform distribution}$$

- **Monotonicity of relative entropy (MONO)** under stochastic matrices N:

$$D(P\|Q) \geq D(N(P)\|N(Q))$$

Entropy for quantum systems

- The **entropy** of $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ is defined as:

$$H(A)_\rho := -\mathrm{tr}\left[\rho_A \log \rho_A\right] = -\sum_x \lambda_x \log \lambda_x \quad \text{[von Neumann 27]}$$

- **Commutative relative entropy** for $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ defined as

$$D_K(\rho\|\sigma) := \sup_{\mathcal{M}} D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)) \quad \text{[Donald 86]}$$

$\mathcal{M} \in Q_{OP}(\mathcal{H} \to \mathcal{H}')$ und $\mathrm{Im}(\mathcal{M}) \subseteq M \subseteq \mathrm{Lin}(\mathcal{H}')$, M commutative subalgebra.

- Some alternative **non-commutative extensions** of $D(P\|Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)}$:

$$D(\rho\|\sigma) := \mathrm{tr}\left[\rho\left(\log \rho - \log \sigma\right)\right] \quad \text{[Umegaki 62]}$$

$$D_B(\rho\|\sigma) := \mathrm{tr}\left[\rho \log\left(\rho^{1/2}\sigma^{-1}\rho^{1/2}\right)\right] \quad \text{[Belavkin, Staszewski 82]}$$

- **Monotonicity of relative entropy (MONO)** for $\mathcal{N} \in Q_{OP}(\mathcal{H} \to \mathcal{H}')$:

$$D_K(\rho\|\sigma) \geq D_K(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \quad \text{[Lindblad 75]}$$

$$D_B(\rho\|\sigma) \geq D_B(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \quad \text{[Fujii, Kamei 89]}$$

Entropy for quantum systems

Quantum relative entropy

For $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with $\rho, \sigma > 0$ we have

$$D_K(\rho\|\sigma) \leq D(\rho\|\sigma) \leq D_B(\rho\|\sigma) \quad \text{with equality if and only if } [\rho, \sigma] = 0.$$

- Access trough **variational formulas** (Legendre transformations):

$$D_K(\rho\|\sigma) = \sup_{\omega>0} \mathrm{tr}\,[\rho \log \omega] + 1 - \mathrm{tr}\,[\sigma\omega] \quad \text{[Donald 86], [B., Fawzi, Tomamichel 17]}$$

$$D(\rho\|\sigma) = \sup_{\omega>0} \mathrm{tr}\,[\rho \log \omega] + 1 - \mathrm{tr}\,[\exp(\log \sigma + \log \omega)] \quad \text{[Araki 73], [Petz 88]}$$

$$D_B(\rho\|\sigma) = \mathrm{tr}\,[\rho \log \omega] + 1 - \mathrm{tr}\,[\sigma \# \omega] \quad \mathrm{for}\ \omega := \rho^{1/2}\sigma^{-1}\rho^{1/2} \quad \text{[Ando, Hiai 94]}$$

  with $\sigma\#\omega := \sigma^{1/2}\left(\sigma^{-1/2}\omega\sigma^{-1/2}\right)^{1/2}\sigma^{1/2}$ the matrix geometric mean.

- **Golden-Thompson** and complementary inequality

$$\mathrm{tr}[\exp(M_1)\exp(M_2)] \geq \mathrm{tr}\,[\exp(M_1 + M_2)] \geq \mathrm{tr}[\exp(M_1)\#\exp(M_2)]$$

Entropy for quantum systems

- The right extension for applications is Umegaki's $D(\rho\|\sigma)$, intuition comes from the **chain rule** [Petz 92] with SSA:

$$H(AB)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(B)_\rho \quad \text{equivalent to}$$

$$D(\rho_{ABC}\|\tau_A \otimes \rho_{BC}) \geq D(\rho_{AB}\|\tau_A \otimes \rho_B) \quad \text{with } \tau_A = \frac{1_A}{\dim(\mathcal{H}_A)}.$$

- All known mathematical properties from MONO $D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$:

Equality conditions MONO [Petz 86]

Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with $\rho \ll \sigma$ and $\mathcal{N} \in Q_{OP}(\mathcal{H} \to \mathcal{H}')$. Then, we have

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = 0$$

if and only if there exists $\mathcal{R}_{\sigma,\mathcal{N}} \in Q_{OP}(\mathcal{H}' \to \mathcal{H})$ such that

$$\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho) = \rho \quad \text{und} \quad \mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\sigma) = \sigma\,.$$

The quantum operation $\mathcal{R}_{\sigma,\mathcal{N}}$ is not unique but can be chosen independent of ρ.

Entropy inequalities

Strong monotonicity (sMONO) [many recent references]

For the same premises we have $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq$ (1), (2), (3) for

$$(1) := -\int \beta_0(t) \log F\left(\rho, \mathcal{R}^{[t]}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)\right) \mathrm{d}t$$ [Junge, Renner, Sutter, Wilde, Winter 15]

$$(2) := D_K\left(\rho \,\middle\|\, \int \beta_0(t) \mathcal{R}^{[t]}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)\, \mathrm{d}t\right)$$ [Sutter, B., Tomamichel 16]

$$(3) := \limsup_{n\to\infty} \frac{1}{n} D\left(\rho^{\otimes n} \,\middle\|\, \int \beta_0(t) \left(\mathcal{R}^{[t]}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)\right)^{\otimes n} \mathrm{d}t\right)$$ [B., Brandão, Hirche 17]

with the probability distribution $\beta_0(t) := \frac{\pi}{2}\left(\cosh(\pi t) + 1\right)^{-1}$ and

$$\mathcal{R}^{[t]}_{\sigma,\mathcal{N}}(\cdot) := \sigma^{\frac{1+it}{2}} \mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-\frac{1+it}{2}} (\cdot) \mathcal{N}(\sigma)^{-\frac{1-it}{2}}\right) \sigma^{\frac{1-it}{2}} \in Q_{OP}(\mathcal{H}' \to \mathcal{H})$$

- We have $D(\rho\|\sigma) \geq D_K(\rho\|\sigma) \geq -\log F(\rho,\sigma)$ but the regularization in (3) is needed [Fawzi[2] 17], leading to incomparable bounds [B., Brandão, Hirche 17].

Entropy inequalities

- Special case of quantum Markov chains $A - B - C$:

Strong SSA (sSSA) [same references]

For $\rho_{ABC} \in \mathcal{S}\left(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C\right)$ we have for the quantum conditional mutual information (CQMI)

$$\begin{aligned} I(A:C|B)_\rho &:= H(AB)_\rho + H(BC)_\rho - H(ABC)_\rho - H(B)_\rho \\ &\geq D_K\left(\rho_{ABC} \middle\| \int \beta_0(t)\left(\mathcal{I}_A \otimes \mathcal{R}^{[t]}_{B\to BC}\right)(\rho_{AB})\,\mathrm{d}t\right) \end{aligned}$$

with

$$\mathcal{R}^{[t]}_{B\to BC}(\cdot) := \rho_{BC}^{\frac{1+it}{2}}\left(\left(\rho_B^{-\frac{1+it}{2}}(\cdot)\rho_B^{-\frac{1-it}{2}}\right)\otimes 1_C\right)\rho_{BC}^{\frac{1-it}{2}} \in Q_{OP}(\mathcal{H}_B \to \mathcal{H}_B \otimes \mathcal{H}_C)\,.$$

(Moreover, we have the analogue fidelity and quantum relative entropy bounds.)

- Proofs of weaker versions of sSSA [Fawzi, Renner 15] + [many references] $\Rightarrow$ matrix analysis will give tight bounds (all previous work superseded).

Entropy inequalities

- **Proof SSA**: Following [Lieb, Ruskai 73] we have with Klein's inequality

$$\begin{aligned} I(A:C|B)_\rho &= D\left(\rho_{ABC}\| \exp\left(\log\rho_{AB} - \log\rho_B + \log\rho_{BC}\right)\right) \\ &\geq \mathrm{tr}\left[\rho_{ABC} - \exp\left(\log\rho_{AB} - \log\rho_B + \log\rho_{BC}\right)\right] \end{aligned}$$

  and we could conclude SSA if $\mathrm{tr}[\exp(\log\rho_{AB} - \log\rho_B + \log\rho_{BC})] \leq 1$. However, just take Lieb's triple matrix inequality

$$\mathrm{tr}\left[\exp(\log M_1 + \log M_2 + \log M_3)\right] \leq \int_0^\infty \mathrm{tr}\left[M_1 \frac{1}{M_2^{-1}+\lambda} M_3 \frac{1}{M_2^{-1}+\lambda}\right] \mathrm{d}\lambda$$

  with $M_1 := \rho_{AB}$, $M_2 := \rho_B^{-1}$, $M_3 := \rho_{BC}$, and use $\int_0^\infty \mathrm{d}\lambda\, x(x^{-1}+\lambda)^{-2} = 1$. $\square$

- **Proof sSSA**: Start with the variational formula

$$\begin{aligned} &I(A:C|B)_\rho = D\left(\rho_{ABC}\| \exp\left(\log\rho_{AB} - \log\rho_B + \log\rho_{BC}\right)\right) \\ &= \sup_{\omega_{ABC}>0} \mathrm{tr}\left[\rho_{ABC}\log\omega_{ABC}\right] + 1 - \mathrm{tr}\left[\exp\left(\log\rho_{AB} - \log\rho_B + \log\rho_{BC} + \log\omega_{ABC}\right)\right] \end{aligned}$$

  but now **four matrix extension** of Golden-Thompson inequality needed?

Multivariate trace inequalities

- Golden-Thompson $\operatorname{tr}[\exp(H_1 + H_2)] \le \operatorname{tr}[\exp(H_1)\exp(H_2)]$ to multivariate:

Multivariate Golden-Thompson [Sutter, B., Tomamichel 16]

For $\{H_k\}_{k=1}^n$ Hermitian, $p \ge 1$, and $\beta_0(t) := \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}$ we have

$$\log \left\| \exp\left(\sum_{k=1}^n H_k\right) \right\|_p \le \int \beta_0(t) \log \left\| \prod_{k=1}^n \exp\left((1+it)H_k\right) \right\|_p \mathrm{d}t\,.$$

The same even holds for any unitarily invariant norm $|||\cdot|||$ as subsequently shown in [Hiai, Koenig, Tomamichel 17].

- For $n = 3$ and $p = 2$ this relaxes to (Jensen's inequality)

$$\begin{aligned} &\operatorname{tr}[\exp(H_1 + H_2 + H_3)] \\ &\le \int \beta_0(t) \operatorname{tr}\left[\exp(H_1)\exp(\frac{1+it}{2}H_2)\exp(H_3)\exp(\frac{1-it}{2}H_2)\right] \mathrm{d}t \end{aligned}$$

which becomes Lieb's bound $\int_0^\infty \operatorname{tr}\left[\exp(H_1)\frac{1}{\exp(-H_2)+\lambda}\exp(H_3)\frac{1}{\exp(-H_2)+\lambda}\right]\mathrm{d}\lambda$.

Multivariate trace inequalities

- Lie-Trotter expansion $\exp\left(\sum_{k=1}^n \log P_k\right) = \lim_{r\to 0} \left|\prod_{k=1}^n P_k^r\right|^{1/r}$:

Multivariate Araki-Lieb-Thirring [Sutter, B., Tomamichel 16]

For $\{M_k\}_{k=1}^n$ positive, $r \in (0,1]$, $p \geq 1$, and $\beta_r(t) := \frac{\sin(\pi r)}{2r(\cosh(\pi t)+\cos(\pi r))}$ we have

$$\log \left\| \left| \prod_{k=1}^n M_k^r \right|^{1/r} \right\|_p \leq \int \beta_r(t) \log \left\| \prod_{k=1}^n M_k^{1+it} \right\|_p \mathrm{d}t.$$

The same holds for any unitarily invariant norm $|||\cdot|||$ [Hiai, Koenig, Tomamichel 17].

- L'Hôpital's rule yields another multivariate extension:

Multivariate Logarithmic Hiai-Petz [Sutter, B., Tomamichel 17]

For $\{M_k\}_{k=1}^n$ positive and $p > 0$ we have (equality in the limit $p \to 0$)

$$\sum_{k=1}^n \mathrm{tr}\left[M_1 \log M_k\right] \geq \frac{1}{p} \int \beta_0(t) \mathrm{tr}\left[M_1 \log\left(M_n^{\frac{p(1+it)}{2}} \cdots M_2^{\frac{p}{2}} M_1^p M_2^{\frac{p}{2}} \cdots M_n^{\frac{p(1-it)}{2}}\right)\right] \mathrm{d}t.$$

Proofs via complex interpolation theory

- Matrix analysis: [Epstein 73], [Kosaki 97]. QIT: [Beigi 13], [Dupuis 15], [Wilde 15], [Junge, Renner, Sutter, Wilde, Winter 15], [Dupuis, Wilde 16].

Strong Hadamard three line theorem [Hirschman 53]

Let $S := \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$, $g : S \to \mathbb{C}$ be uniformly bounded on S, holomorph in the interior of S, and continous on the boundary. Then, we have for $r \in (0,1)$ with $\beta_r(t) := \frac{\sin(\pi r)}{2r(\cosh(\pi t)+\cos(\pi r))}$ that:

$$\begin{aligned}\log|g(r)| &\leq \int \left(\beta_{1-r}(t)\log|g(it)|^{1-r} + \beta_r(t)\log|g(1+it)|^r\right)\mathrm{d}t \\ &\leq \sup_t \log|g(it)|^{1-r} + \sup_t \log|g(1+it)|^r\end{aligned}$$

- For $G(z)$ matrix valued uniformly bounded holomorphic function we have [Stein]:

$$\log\|G(r)\|_\infty \leq \int \mathrm{d}t\,\left(\beta_{1-r}(t)\log\|G(it)\|_\infty^{1-r} + \beta_r(t)\log\|G(1+it)\|_\infty^r\right)$$

- Proof: For $r \in (0,1)$ let $u, v \in \mathbb{C}^d$ be normalized s.t. $\|G(r)\|_\infty = \langle u, G(r)v\rangle$ and hence for $g(z) := \langle u, G(z)v\rangle$ we get $|g(z)| \leq \|G(z)\|_\infty$ for all $z \in S$. $\square$

Proofs via complex interpolation theory

- **Multivariate Araki-Lieb-Thirring** for $|||\cdot||| = \|\cdot\|_\infty$ with $r \in (0,1]$ and $M_k > 0$:

$$\log \left\| \left| \prod_{k=1}^{n} M_k^r \right|^{1/r} \right\|_\infty \leq \int \beta_r(t) \log \left\| \prod_{k=1}^{n} M_k^{1+it} \right\|_\infty \mathrm{d}t$$

- Proof: Apply Stein-Hirschman

$$\log \|G(r)\|_\infty \leq \int \mathrm{d}t \, \left(\beta_{1-r}(t) \log \|G(it)\|_\infty^{1-r} + \beta_r(t) \log \|G(1+it)\|_\infty^{r} \right)$$

for the function

$$G(z) := \prod_{k=1}^{n} M_k^z = \prod_{k=1}^{n} \exp(z \log M_k)$$

and since $M_k > 0$, M_k^{it} becomes unitary, and hence $\log \|\cdot\|_\infty^{1-r} = 0$. $\square$

- General case for unitarily invariant norms via anti-symmetric tensor product (or for Schatten p-norms with more general Stein-Hirschman interpolation).
- **Multivariate Golden-Thompson** from Lie-Trotter expansion.

Proof of entropy inequalities

- For the proof of sSSA we choose $n = 4$, $|||\cdot||| = \|\cdot\|_2$ and we get for $M_1, M_2, M_3, M_4 > 0$ (Lieb's triple matrix inequality corresponds to $n = 3$):

$$\mathrm{tr}\left[\exp\left(\sum_{i=1}^{4} \log M_i\right)\right] \leq \int \beta_0(t)\mathrm{tr}\left[M_1 M_2^{\frac{1+it}{2}} M_3^{\frac{1+it}{2}} M_4 M_3^{\frac{1-it}{2}} M_2^{\frac{1-it}{2}}\right] \mathrm{d}t$$

- Choose $M_1 := \omega_{ABC}$, $M_2 := \rho_{BC}$, $M_3 := \rho_B^{-1}$, $M_4 := \rho_{AB}$ and thus

$$\begin{aligned}
I(A:C|B)_\rho &= D\left(\rho_{ABC} \| \exp\left(\log \rho_{AB} - \log \rho_B + \log \rho_{BC}\right)\right) \\
&= \sup_{\omega_{ABC}>0} \mathrm{tr}\left[\rho_{ABC} \log \omega_{ABC}\right] + 1 - \mathrm{tr}\left[\exp\left(\log \rho_{AB} - \log \rho_B + \log \rho_{BC} + \log \omega_{ABC}\right)\right] \\
&\geq \sup_{\omega_{ABC}>0} \mathrm{tr}\left[\rho_{ABC} \log \omega_{ABC}\right] + 1 - \int \beta_0(t)\mathrm{tr}\left[\omega_{ABC} \rho_{BC}^{\frac{1+it}{2}} \rho_B^{-\frac{1+it}{2}} \rho_{AB} \rho_B^{-\frac{1-it}{2}} \rho_{BC}^{\frac{1+it}{2}}\right] \mathrm{d}t \\
&\geq D_K\left(\rho_{ABC} \middle\| \int \mathrm{d}t\, \beta_0(t) \rho_{BC}^{\frac{1+it}{2}} \rho_B^{-\frac{1+it}{2}} \rho_{AB} \rho_B^{-\frac{1-it}{2}} \rho_{BC}^{\frac{1+it}{2}}\right) \qquad \square
\end{aligned}$$

- All known sSSA/sMONO bounds follow from multivariate Golden-Thompson.

Conclusion

- Strengthened entropy inequalities sSSA/sMONO through multivariate matrix trace inequalities from complex interpolation theory
- Different **representation of integral** in, e.g.,

$$\mathrm{tr}\left[\exp\left(\sum_{i=1}^{3} H_i\right)\right] \leq \int \beta_0(t)\mathrm{tr}\left[\exp(H_1)\exp(\frac{1+it}{2}H_2)\exp(H_3)\exp(\frac{1-it}{2}H_2)\right]\mathrm{d}t$$

as explored in [Lemm 17], [Garg, Lee, Song, Srivastava 17]:

$$\int \beta_0(t)\cdots\mathrm{d}t \quad \text{vs.} \quad \int_0^\infty \cdots\mathrm{d}\lambda \quad \text{vs.} \quad \frac{4}{\pi}\int_{-\pi/2}^{\pi/2}\cdots\mathrm{d}\mu(\phi).$$

- **Multivariate matrix analysis?** Complementary trace inequalities [Hiai, Petz 08]:

$$\mathrm{tr}[\exp(M_1)\exp(M_2)] \geq \mathrm{tr}\left[\exp(M_1+M_2)\right] \geq \mathrm{tr}[\exp(M_1)\#\exp(M_2)]$$

$\Rightarrow$ development in MathPhys, e.g., [Carlen, Lieb 17], [Carlen, Vershynina 17].
- Virtually any (mathematical) property of quantum entropy follows from *n*-extension of Golden-Thompson inequality!

Outlook – novel quantum entropy inequalities?

- From [Christandl, Brandão, Yard 11] + [Li, Winter 14] we have

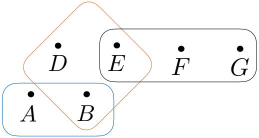
$$I(A:C|B)_\rho \geq \inf_{\sigma \in \mathrm{SEP}(A:C)} D_{\mathrm{LOCC}(1)}(\rho_{AC} \| \sigma_{AC}) \quad \text{with}$$

  the local operation and classical communication measured relative entropy

$$D_{\mathrm{LOCC}(1)}(\rho_{AC} \| \sigma_{AC}) := \sup_{\mathcal{M} \in \mathrm{LOCC}(1)} D(\mathcal{M}(\rho_{AC}) \| \mathcal{M}(\sigma_{AC}))$$

  $\Rightarrow$ no direct relation/proof [Christandl 15] + extensions conjectured [Brandão 17].
- Entropy inequalities in the other direction – upper bounds on CQMI?
- Non-Shannon type information inequalities for entropy cone [Pippenger 03]:

$$3H(AB)_\rho + H(AC)_\rho + 3H(AD)_\rho + 3H(BD)_\rho + H(CD)_\rho \geq 2H(A)_\rho + 4H(ABD)_\rho + H(ACD)_\rho + H(B)_\rho + 2H(D)_\rho.$$



- Quantum information science [references]: novel **constraints on multipartite entanglement structure**