Exploiting Variational Formulas for Quantum Relative Entropy

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joint work with Omar Fawzi and Marco Tomamichel

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Overview

Classical Relative Entropy

Definition (Relative Entropy)

For a positive measure Q on a finite set $\mathcal{X}$ and a probability measure P on $\mathcal{X}$ with $P \ll Q$, the relative entropy is defined as

$$D(P\|Q) := \sum_{x\in\mathcal{X}} P(x)\log\frac{P(x)}{Q(x)} \quad \text{[Kullback-Leibler 1951]}\,, \tag{1}$$

where we understand $P(x)\log\frac{P(x)}{Q(x)} = 0$ whenever $P(x) = 0$.

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For $\alpha \in (0,1)\cup(1,\infty)$ the Rényi relative entropy is defined as

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- What are the quantum extensions of $D(P\|Q)$ and $D_\alpha(P\|Q)$?

Measured Relative Entropy

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On a Hilbert space $\mathcal{H}$, for quantum states ρ and positive semi-definite operators σ the measured relative entropy is defined as

$$D^{\mathbb{M}}(\rho\|\sigma) := \sup_{(\mathcal{X},M)} D\big(P_{\rho,M}\big\|P_{\sigma,M}\big) \quad \text{[Donald 1986]}\,, \tag{3}$$

where the optimization is over finite sets $\mathcal{X}$ and positive operator valued measures (POVMs) M on $\mathcal{X}$, and $P_{\rho,M}(x) = \mathrm{tr}[M(x)\rho]$.

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- Are there other quantum extensions?

Quantum Relative Entropy

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For the same setup as before, the quantum relative entropy is defined as

$$D(\rho\|\sigma) := \mathrm{tr}\big[\rho(\log\rho - \log\sigma)\big] \quad \text{[Umegaki 1962]} \tag{5}$$

if $\sigma \gg \rho$ and $+\infty$ if $\sigma \not\gg \rho$.

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[Müller-Lennert *et al.* 2013, Wilde *et al.* 2015] . (7)

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- How is $D^{\mathbb{M}}(\rho\|\sigma)$ related to $D(\rho\|\sigma)$? How is $D^{\mathbb{M}}_\alpha(\rho\|\sigma)$ related to $D_\alpha(\rho\|\sigma)$?

Variational Formulas for Relative Entropy I

- Variational Formula for Quantum Relative Entropy: [Petz 1988] we have

$$D(\rho\|\sigma) = \sup_{\omega>0} \operatorname{tr}[\rho \log \omega] + 1 - \operatorname{tr}[\exp(\log \sigma + \log \omega)]\,. \tag{8}$$

Theorem (Variational Formula for Measured Relative Entropy)

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- By the Golden-Thompson inequality $\operatorname{tr}\left[\log \sigma + \log \omega\right] \leq \operatorname{tr}[\sigma\omega]$ we find:

Theorem (Achievability of Quantum Relative Entropy)

We have

$$D^{\mathbb{M}}(\rho\|\sigma) \leq D(\rho\|\sigma) \quad \textit{with equality if and only if } [\rho,\sigma] = 0. \tag{10}$$

Variational Formulas for Relative Entropy II

- Variational formula for sandwiched Rényi relative entropy: [Frank & Lieb 2013] for $Q_\alpha(\rho\|\sigma) = \exp\left((\alpha-1)D_\alpha(\rho\|\sigma)\right)$ we have

$$Q_\alpha(\rho\|\sigma) = \begin{cases} \inf\limits_{\omega>0} \alpha\,\mathrm{tr}[\rho\omega] + (1-\alpha)\,\mathrm{tr}\left[\left(\omega^{\frac{1}{2}}\sigma^{\frac{\alpha-1}{\alpha}}\omega^{\frac{1}{2}}\right)^{\frac{\alpha}{\alpha-1}}\right] & \text{for } \alpha\in(0,1) \\ \sup\limits_{\omega>0} \alpha\,\mathrm{tr}[\rho\omega] + (1-\alpha)\,\mathrm{tr}\left[\left(\omega^{\frac{1}{2}}\sigma^{\frac{\alpha-1}{\alpha}}\omega^{\frac{1}{2}}\right)^{\frac{\alpha}{\alpha-1}}\right] & \text{for } \alpha\in(1,\infty)\,. \end{cases} \tag{11}$$

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Theorem (Variational Formula for Measured Rényi Relative Entropy)

For $Q_\alpha^{\mathbb{M}}(\rho\|\sigma) := \exp\left((\alpha-1)D_\alpha^{\mathbb{M}}(\rho\|\sigma)\right)$ *we have*

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By the Araki-Lieb-Thirring inequality we have for $\alpha \in (1/2,\infty)$*:*

$$D_\alpha^{\mathbb{M}}(\rho\|\sigma) \le D_\alpha(\rho\|\sigma) \quad \text{with equality if and only if } [\rho,\sigma]=0. \tag{13}$$

Application: Additivity in Quantum Information I

- We consider operational quantities of the form

$$\mathcal{M}(\rho) := \min_{\sigma\in\mathcal{C}} \mathbb{D}(\rho\|\sigma)\,, \tag{14}$$

where $\mathbb{D}(\,\cdot\,\|\,\cdot\,)$ stands for any relative entropy, and $\mathcal{C}$ denotes some convex, compact set of states.

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  - separable states: relative entropy of entanglement [Vedral 1998]
  - positive partial transpose states: Rains bound on entanglement distillation [Rains 2001]
  - non-distillable states: bounds on entanglement distillation [Vedral 1999]
  - quantum Markov states: robustness properties of these states [Linden *et al.* 2008]
  - locally recoverable states: bounds on the conditional mutual information [Fawzi & Renner 2015]
  - k-extendible states: bounds on squashed entanglement [Li & Winter 2014]

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- Question: What properties of the relative entropy translate to the measure $\mathcal{M}$?
- For example, all the relative entropies discussed are super-additive on tensor product states

$$\mathbb{D}(\rho_1 \otimes \rho_2\|\sigma_1 \otimes \sigma_2) \geq \mathbb{D}(\rho_1\|\sigma_1) + \mathbb{D}(\rho_2\|\sigma_2)\,. \tag{15}$$

Application: Additivity in Quantum Information II

- Super-additivity of $\mathcal{M}$ on tensor product states:

$$\min_{\sigma_{12}\in\mathcal{C}_{12}} \mathbb{D}(\rho_1 \otimes \rho_2 \| \sigma_{12}) = \mathcal{M}(\rho_1 \otimes \rho_2) \overset{?}{\geq} \mathcal{M}(\rho_1) + \mathcal{M}(\rho_2) \tag{16}$$

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- Idea: Use variational characterizations

$$\mathbb{D}(\rho\|\sigma) = \sup_{\omega>0} f(\rho,\sigma,\omega) \text{ in order to write} \tag{18}$$

$$\mathcal{M}(\rho) = \min_{\sigma\in\mathcal{C}} \sup_{\omega>0} f(\rho,\sigma,\omega) = \sup_{\omega>0} \min_{\sigma\in\mathcal{C}} f(\rho,\sigma,\omega)\,, \tag{19}$$

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  where we made use of Sion's minimax theorem for the last equality.
- The minimization over $\sigma \in \mathcal{C}$ then often simplifies and is a convex or even semidefinite optimization. Using strong duality to rewrite this minimization as a maximization problem:

$$\min_{\sigma\in\mathcal{C}} f(\rho,\sigma,\omega) = \max_{\bar{\sigma}\in\bar{\mathcal{C}}} \bar{f}(\rho,\bar{\sigma},\omega) \text{ leading to the expression} \tag{20}$$

$$\mathcal{M}(\rho) = \sup_{\omega>0} \max_{\bar{\sigma}\in\bar{\mathcal{C}}} \bar{f}(\rho,\bar{\sigma},\omega)\,. \tag{21}$$

Application: Additivity in Quantum Information III

- Variational characterization (as on the last slide):

$$\mathcal{M}(\rho) = \sup_{\omega>0} \max_{\bar{\sigma}\in\bar{\mathcal{C}}} \bar{f}(\rho,\bar{\sigma},\omega)\,. \tag{22}$$

Application: Additivity in Quantum Information III

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$$\mathcal{M}(\rho) = \sup_{\omega > 0} \max_{\bar{\sigma} \in \bar{\mathcal{C}}} \bar{f}(\rho, \bar{\sigma}, \omega) \,. \tag{22}$$

- The following two conditions on $\bar{f}$ and $\bar{\mathcal{C}}$ imply super-additivity of $\mathcal{M}$:
  1. $\bar{f}$ is super-additive

$$\bar{f}(\rho_1 \otimes \rho_2, \bar{\sigma}_1 \otimes \bar{\sigma}_2, \omega_1 \otimes \omega_2) \geq \bar{f}(\rho_1, \bar{\sigma}_1, \omega_1) + \bar{f}(\rho_2, \bar{\sigma}_2, \omega_2) \tag{23}$$

  2. the sets $\bar{\mathcal{C}}$ are closed under tensor products

$$\bar{\sigma}_1 \in \bar{\mathcal{C}}_1 \text{ and } \bar{\sigma}_2 \in \bar{\mathcal{C}}_2 \quad \text{imply that} \quad \bar{\sigma}_1 \otimes \bar{\sigma}_2 \in \bar{\mathcal{C}}_{12} \tag{24}$$

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- Proof: For any $\omega_1, \omega_2 > 0$ and any $\bar{\sigma}_1 \in \bar{\mathcal{C}}_1$, $\bar{\sigma}_2 \in \bar{\mathcal{C}}_2$ we deduce that

$$\mathcal{M}(\rho_1 \otimes \rho_2) \geq \bar{f}(\rho_1 \otimes \rho_2, \bar{\sigma}_1 \otimes \bar{\sigma}_2, \omega_1 \otimes \omega_2) \geq \bar{f}(\rho_1, \bar{\sigma}_1, \omega_1) + \bar{f}(\rho_2, \bar{\sigma}_2, \omega_2) \tag{25}$$

Hence, the inequalities also hold true if we maximize over these variables, implying super-additivity. □

Example: Relative Entropy of Recovery I

- For any relative entropy $\mathbb{D}(\cdot\|\cdot)$ we are interested in the recovery quantity:

$$\mathbb{D}^{\text{rec}}(\rho_{AD}\|\sigma_{AE}) := \inf_{\Gamma_{E\to D}} \mathbb{D}(\rho_{AD}\|(\mathcal{I}_A\otimes\Gamma_{E\to D})(\sigma_{AE}))\,. \tag{26}$$

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- Question: What relative entropies of recovery are (super)-additive?
- Of interest because we have (the systems are understood as $D = BC$, $E = B$):

$$I(A:C|B) \geq \lim_{n\to\infty} \frac{1}{n} D^{\mathrm{rec}}\left(\rho_{ABC}^{\otimes n}\|\rho_{AB}^{\otimes n}\right) \quad \text{[Brandao \textit{et al.} 2015]}\,. \tag{27}$$

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Theorem (Super-Additivity of Measured Entropy of Recovery)

Let $\rho_{AD}, \tau_{A'D'}, \sigma_{AE}, \omega_{A'E'}$ be quantum states. Then, we have

$$D^{\mathbb{M},\mathrm{rec}}(\rho_{AD}\otimes\tau_{A'D'}\|\sigma_{AE}\otimes\omega_{A'E'}) \geq D^{\mathbb{M},\mathrm{rec}}(\rho_{AD}\|\sigma_{AE}) + D^{\mathbb{M},\mathrm{rec}}(\tau_{A'D'}\|\omega_{A'E'})\,. \tag{28}$$

For $\alpha \in (0,1)\cup(1,\infty)$, we also have

$$D_\alpha^{\mathbb{M},\mathrm{rec}}(\rho_{AD}\otimes\tau_{A'D'}\|\sigma_{AE}\otimes\omega_{A'E'}) \geq D_\alpha^{\mathbb{M},\mathrm{rec}}(\rho_{AD}\|\sigma_{AE}) + D_\alpha^{\mathbb{M},\mathrm{rec}}(\tau_{A'D'}\|\omega_{A'E'})\,. \tag{29}$$

Example: Relative Entropy of Recovery II

- We conclude (without de Finetti reductions):

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- The super-additivity is seen be the following dual variational characterization:

Lemma (Variational Representation of Measured Entropy of Recovery)

Let ρ_{AD}, σ_{AE} be quantum states, and let σ_{AEF} be a purification of σ_{AE}. Then, we have

$$\begin{aligned} D^{\mathbb{M},\mathrm{rec}}(\rho_{AD} \| \sigma_{AE}) = \text{maximize:} \quad & \mathrm{tr}[\rho_{AD} \log R_{AD}] \\ \text{subject to:} \quad & S_{AF} > 0,\ R_{AD} > 0 \\ & 1_D \otimes S_{AF} \geq R_{AD} \otimes 1_F \\ & \mathrm{tr}[S_{AF}\sigma_{AF}] = 1 \, . \end{aligned} \qquad (31)$$

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- The characterization for the measured Rényi entropy of recovery $D_\alpha^{\mathbb{M},\text{rec}}$ is similar.
- The relative entropy of recovery D^{rec} and the Rényi entropy of recovery D_α^{rec} do not seem to be additive (but open).

Example: Relative Entropy of Recovery III

Lemma (Variational Representation of Entropy of Recovery)

For the same premises as before we have

$$\begin{aligned} D^{\mathrm{rec}}(\rho_{AD}\|\sigma_{AE}) = \text{maximize:} \quad & \mathrm{tr}[\rho_{AD}\log\rho_{AD}] - D^{\mathbb{M}}(\rho_{AD}\|R_{AD}) \\ \text{subject to:} \quad & S_{AF} > 0,\ R_{AD} > 0 \\ & 1_D \otimes S_{AF} \geq R_{AD} \otimes 1_F \\ & \mathrm{tr}[S_{AF}\sigma_{AF}] = 1\,. \end{aligned} \tag{32}$$

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Lemma (Variational Representation of Entropy of Recovery)

For the same premises as before we have

$$\begin{aligned} D^{\mathrm{rec}}(\rho_{AD}\|\sigma_{AE}) = \text{maximize:} \quad & \mathrm{tr}[\rho_{AD}\log\rho_{AD}] - D^{\mathbb{M}}(\rho_{AD}\|R_{AD}) \\ \text{subject to:} \quad & S_{AF} > 0,\ R_{AD} > 0 \\ & 1_D \otimes S_{AF} \geq R_{AD} \otimes 1_F \\ & \mathrm{tr}[S_{AF}\sigma_{AF}] = 1\,. \end{aligned} \tag{32}$$

- This is to be compared to the dual characterization of the measured entropy of recovery $D^{\mathbb{M},\mathrm{rec}}(\rho_{AD}\|\sigma_{AE})$:

$$\begin{aligned} D^{\mathbb{M},\mathrm{rec}}(\rho_{AD}\|\sigma_{AE}) = \text{maximize:} \quad & \mathrm{tr}[\rho_{AD}\log\rho_{AD}] - D(\rho_{AD}\|R_{AD}) \\ \text{subject to:} \quad & S_{AF} > 0,\ R_{AD} > 0 \\ & 1_D \otimes S_{AF} \geq R_{AD} \otimes 1_F \\ & \mathrm{tr}[S_{AF}\sigma_{AF}] = 1\,. \end{aligned} \tag{33}$$

Conclusion

- Measured relative entropy is strictly smaller than quantum relative entropy.
- Variational formulas for quantum relative entropy.
- These formulas are useful tools for studying additivity problems in quantum information theory.
- Super-additivity of measured relative entropy of recovery.
- Additivity of relative entropy of recovery remains open (does not seem to be additive).