The Quantum Reverse Shannon Theorem and other Channel Simulations

Mario Berta (Fernando Brandão, Matthias Christandl, Renato Renner, Stephanie Wehner)

Quantum Reverse Shannon Theorem

- Previously proved by Bennett, Devetak, Harrow, Shor and Winter [1].
- New proof based on one-shot Quantum State Merging [2,3] and the Post-Selection Technique for Quantum Channels [4].
- Outline:
  - Understanding the Theorem (Classical and Quantum Shannon Theory)
  - Idea of our Proof
  - Quantum State Merging
  - Post-Selection Technique
  - Other Channel Simulations

[1] arXiv.org/quant-ph:0912.5537
[2] Horodecki et al., Nature 436:673-676, 2005
[3] Berta, arXiv.org/quant-ph:0912.4495
[4] Christandl et al., Phys. Rev. Lett. 102:020504, 2009

Shannon's Classical Noisy Channel Coding Theorem

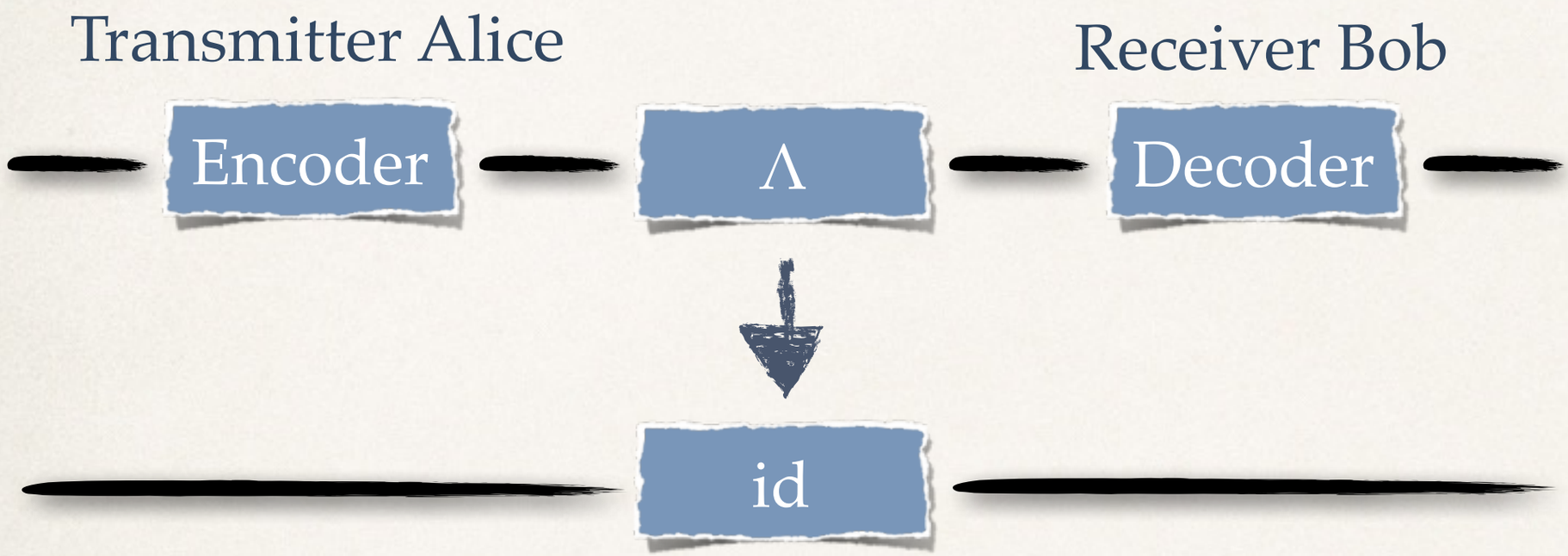
Transmitter Alice

Receiver Bob

Λ: noisy channel

How many bits can Alice transmit on average per use of the channel?

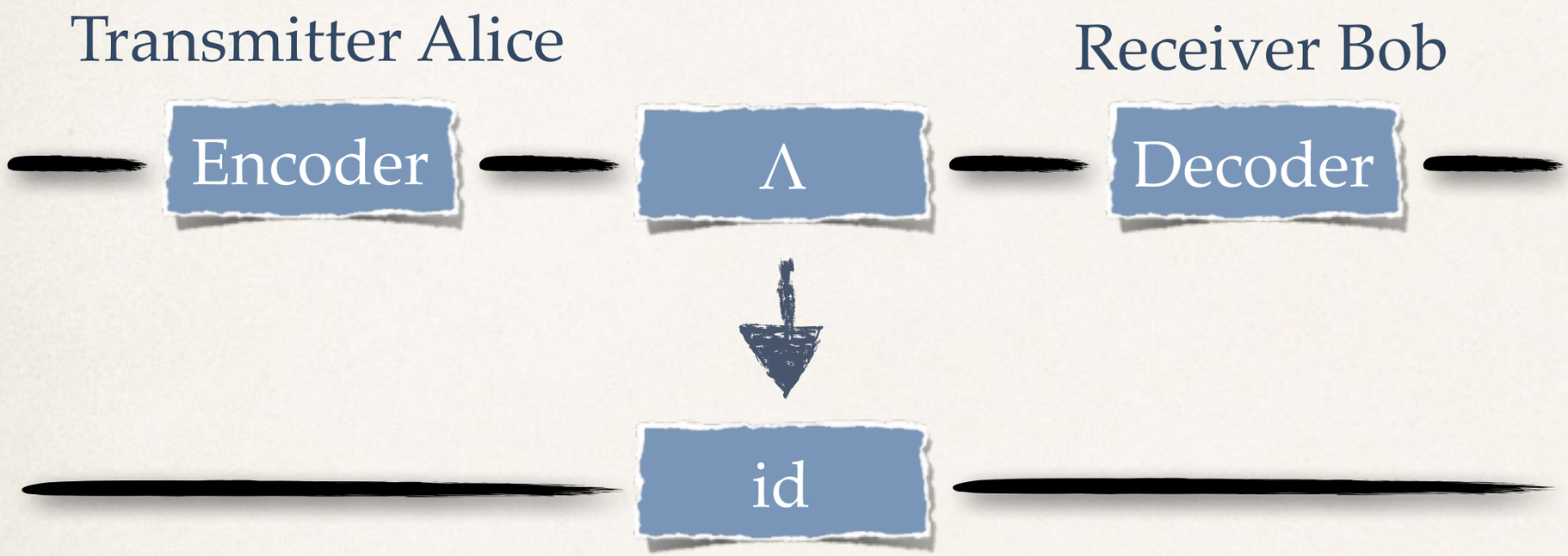
Shannon's Classical Noisy Channel Coding Theorem



Λ: noisy channel
id: perfect channel

How many bits can Alice transmit on average per use of the channel?

Shannon's Classical Noisy Channel Coding Theorem



Λ: noisy channel
id: perfect channel

How many bits can Alice transmit on average per use of the channel?

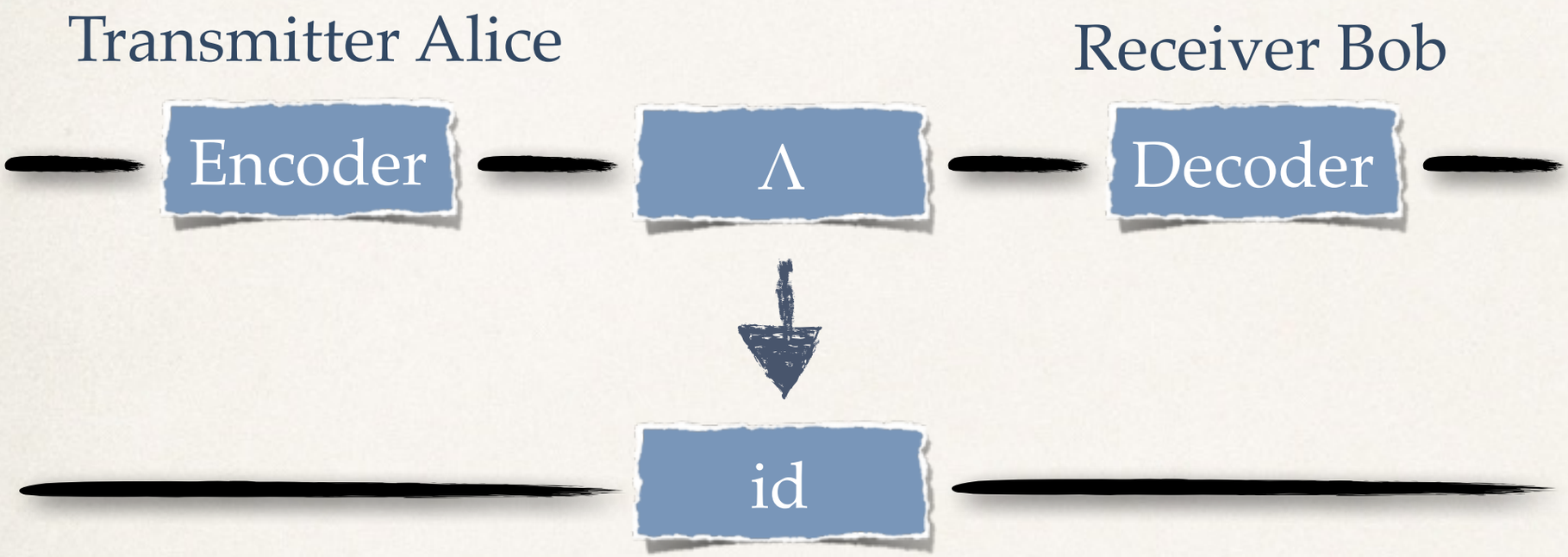
$\Rightarrow$ Asymptotic channel capacity [5]:

$$C(\Lambda) = \max_X (H(X) + H(\Lambda(X)) - H(X, \Lambda(X)))$$

$$H(X) = -\sum_x p_x \log p_x$$

[5] Shannon, Bell. Syst. Tech. J. 27:379-423,623-656, 1948

Shannon's Classical Noisy Channel Coding Theorem



Λ: noisy channel
id: perfect channel

How many bits can Alice transmit on average per use of the channel?

$\Rightarrow$ Asymptotic channel capacity [5]:

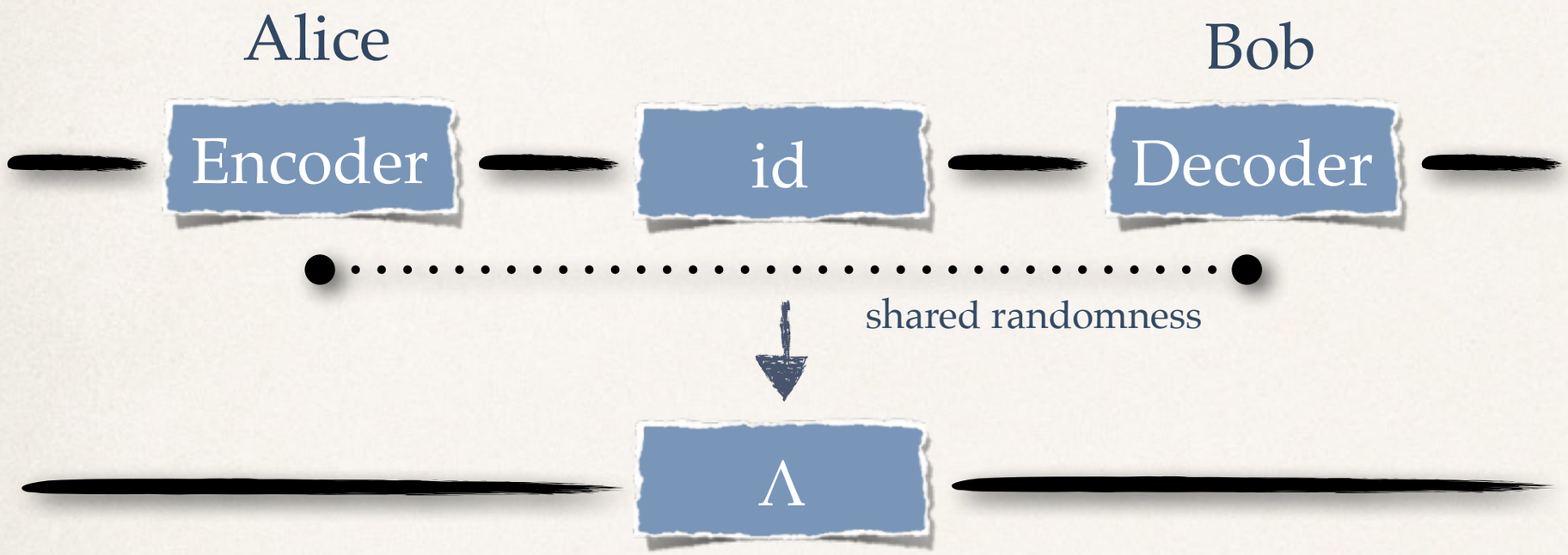
$$C(\Lambda) = \max_X (H(X) + H(\Lambda(X)) - H(X, \Lambda(X)))$$

$$H(X) = -\sum_x p_x \log p_x$$

Note: Neither back communication nor shared randomness help

[5] Shannon, Bell. Syst. Tech. J. 27:379-423,623-656, 1948

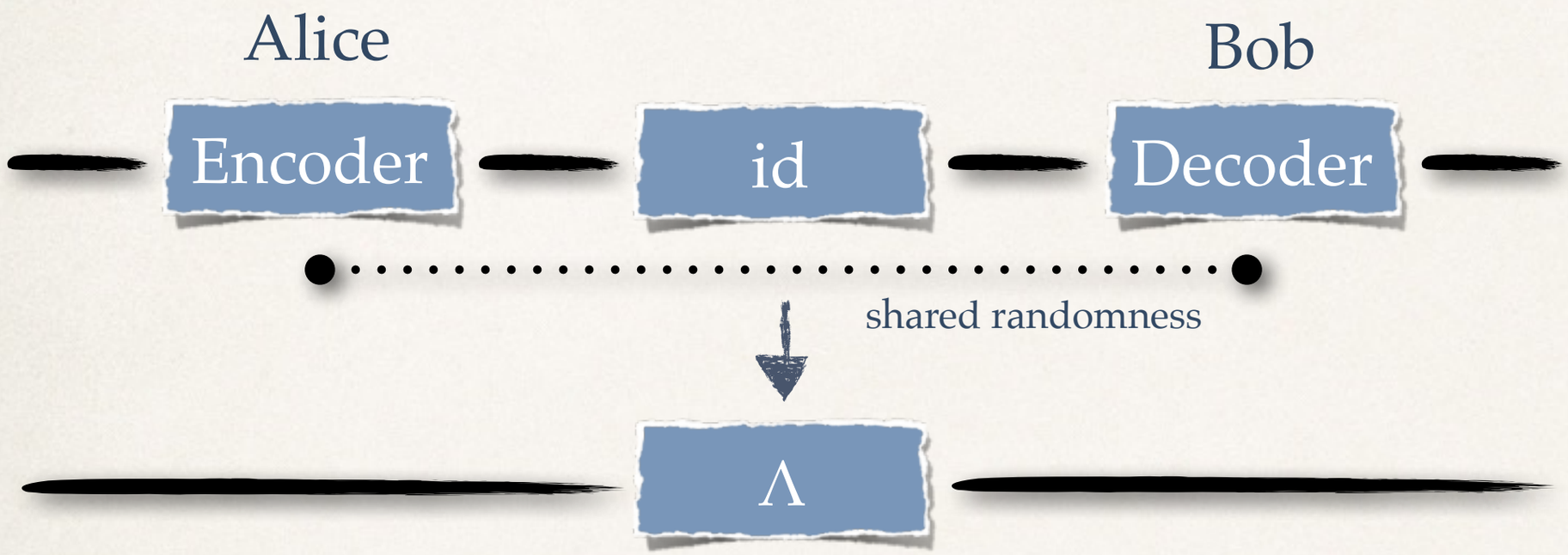
Classical Reverse Shannon Theorem



id: perfect channel
Λ: noisy channel

Using shared randomness, at what asymptotic rate can the id-channel simulate a channel Λ?

Classical Reverse Shannon Theorem



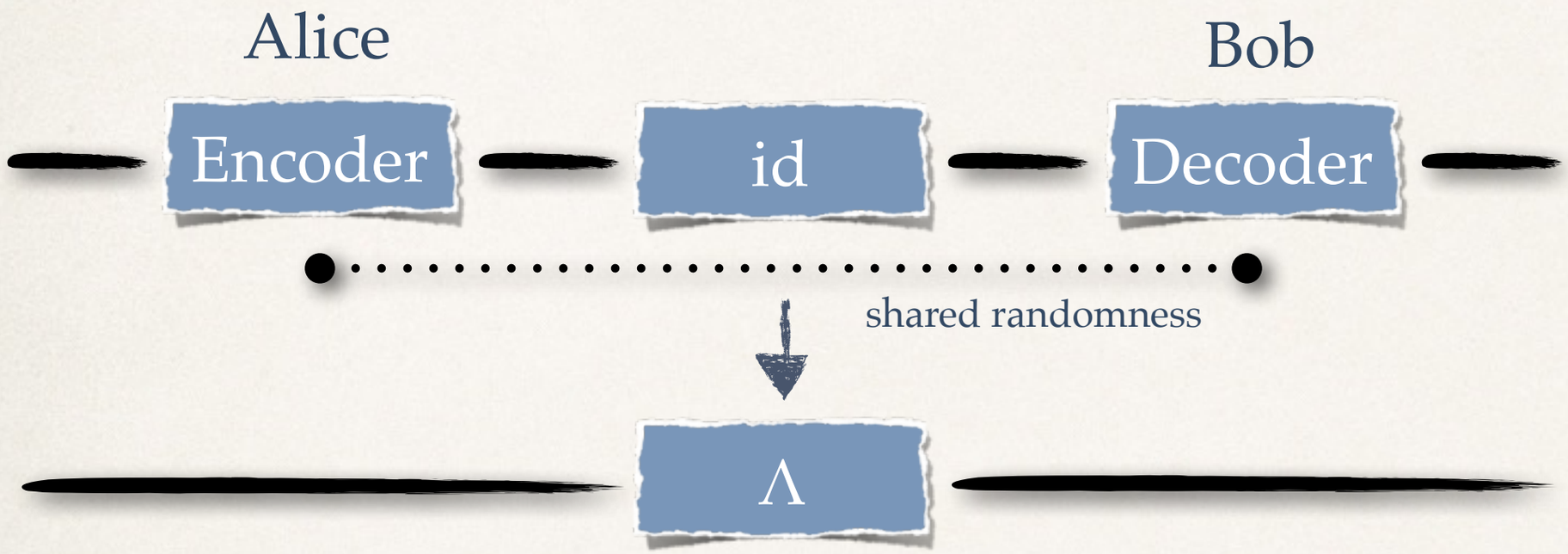
id: perfect channel
Λ: noisy channel

Using shared randomness, at what asymptotic rate can the id-channel simulate a channel Λ?

⇒ C(Λ) as well [6]!

[6] Bennett et al., IEEE Trans. Inf. Theory 48(10):2637, 2002

Classical Reverse Shannon Theorem

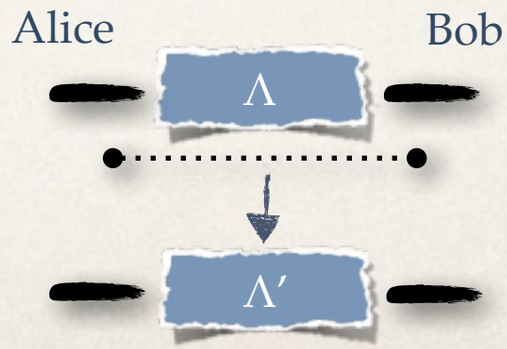


id: perfect channel
Λ: noisy channel

Using shared randomness, at what asymptotic rate can the id-channel simulate a channel Λ?

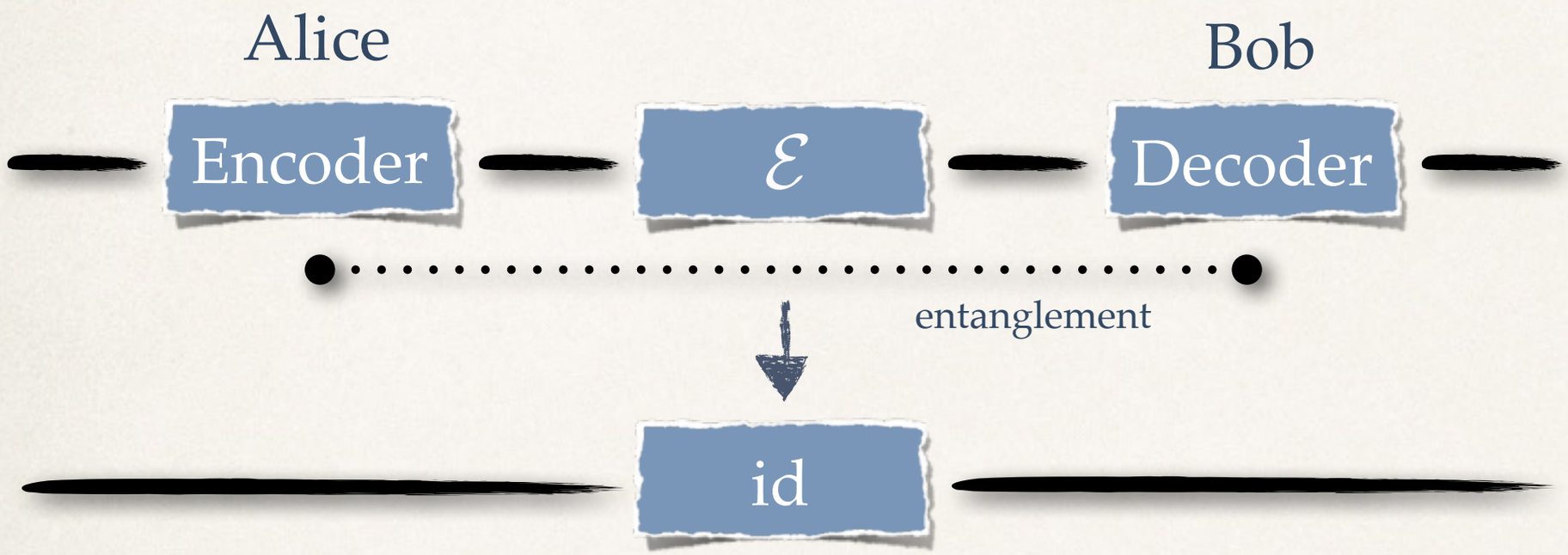
⇒ C(Λ) as well [6]! I.e. the asymptotic capacity of a channel Λ to simulate another channel Λ′ in the presence of free shared randomness is given by:

$$C_R(\Lambda, \Lambda') = \frac{C(\Lambda)}{C(\Lambda')}$$



[6] Bennett et al., IEEE Trans. Inf. Theory 48(10):2637, 2002

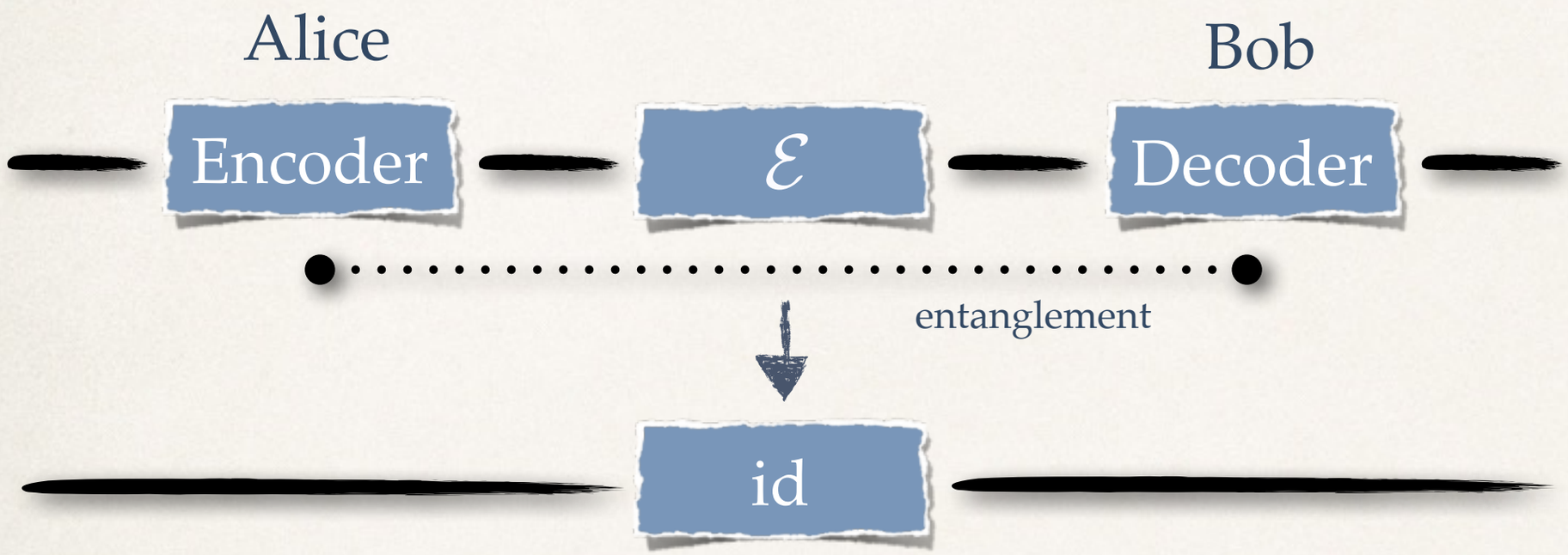
Quantum Shannon Theorem



$\mathcal{E}$: quantum channel
id: perfect classical channel

Using entanglement, at what asymptotic rate can Alice transmit classical information?

Quantum Shannon Theorem



$\mathcal{E}$: quantum channel
id: perfect classical channel

Using entanglement, at what asymptotic rate can Alice transmit classical information?

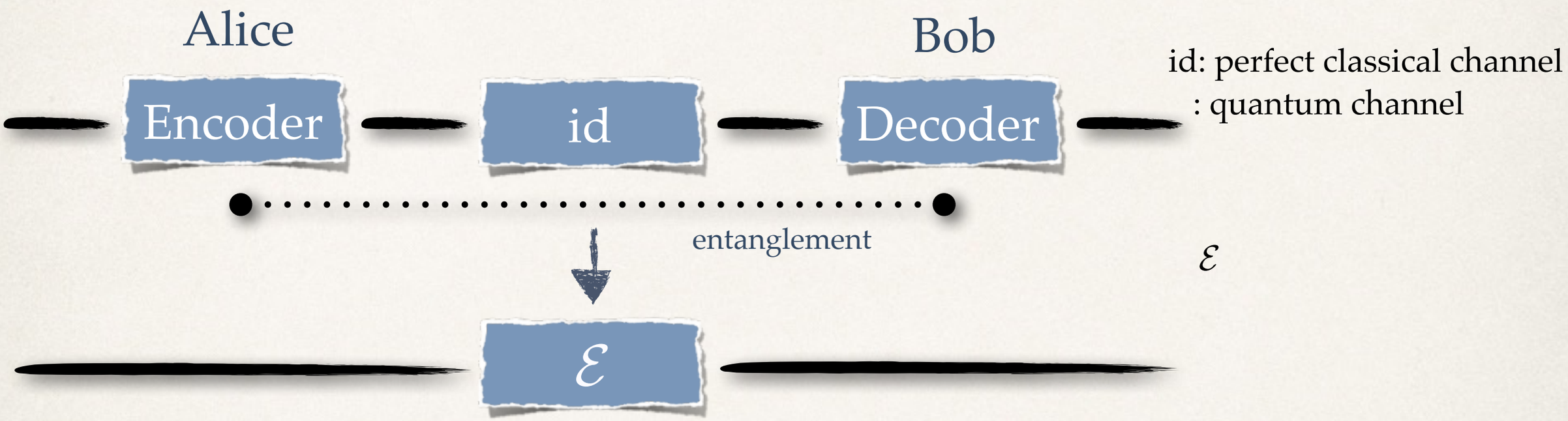
$\Rightarrow$ Asymptotic entanglement-assisted classical capacity [6]:

$$C_E = \max_{\rho}(H(\rho) + H(\mathcal{E}(\rho)) - H((\mathcal{E} \otimes \mathrm{id})\Phi_\rho))$$

$$H(\rho) = -\mathrm{tr}(\rho \log \rho)$$

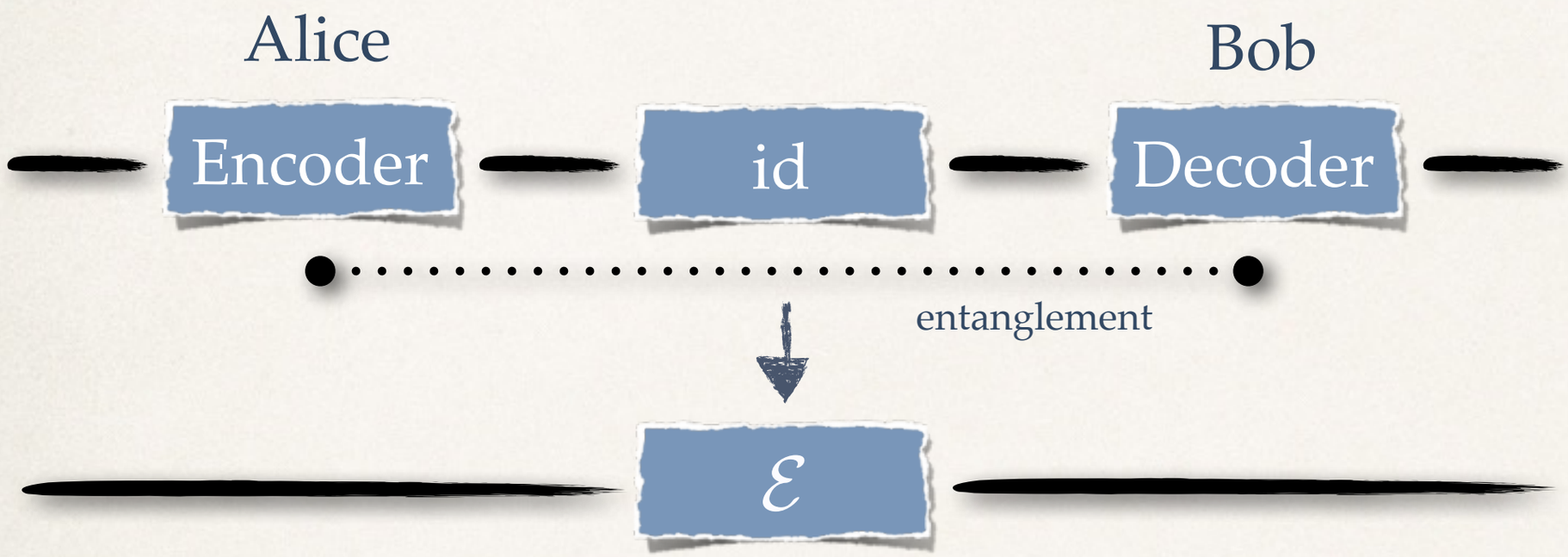
[6] Bennett et al., IEEE Trans. Inf. Theory 48(10):2637, 2002

Quantum Reverse Shannon Theorem



Using entanglement, at what asymptotic rate can the classical id-channel simulate a quantum channel?

Quantum Reverse Shannon Theorem

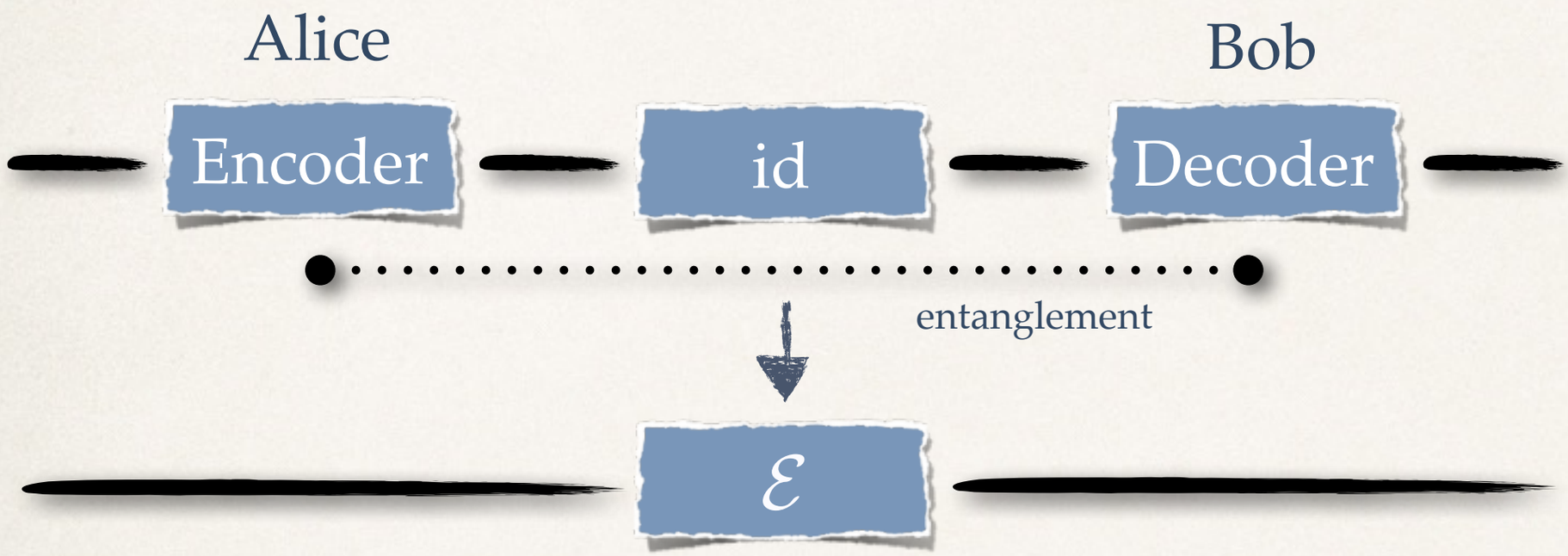


id: perfect classical channel
$\mathcal{E}$: quantum channel

Using entanglement, at what asymptotic rate can the classical id-channel simulate a quantum channel?

$\Rightarrow C_E$ as well!

Quantum Reverse Shannon Theorem

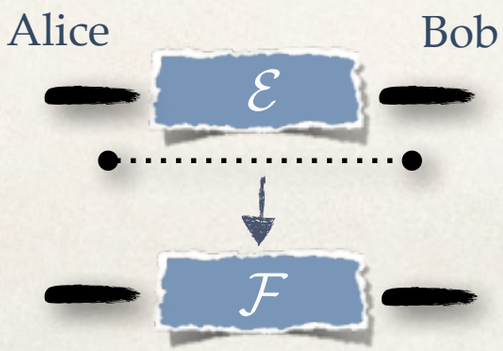


id: perfect classical channel
$\mathcal{E}$: quantum channel

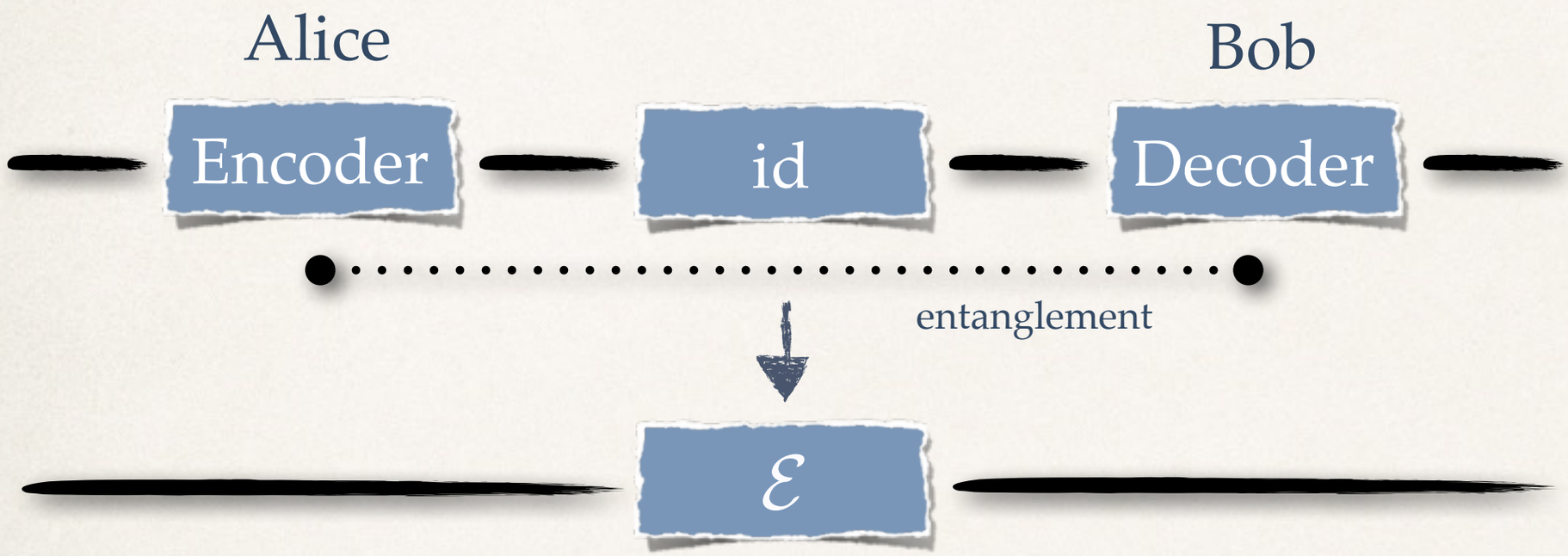
Using entanglement, at what asymptotic rate can the classical id-channel simulate a quantum channel?

⇒ C_E as well! I.e. the asymptotic capacity of a quantum channel to simulate another quantum channel in the presence of free entanglement is given by:

$$C_E(\mathcal{E},\mathcal{F}) = \frac{C_E(\mathcal{E})}{C_E(\mathcal{F})}$$



Quantum Reverse Shannon Theorem

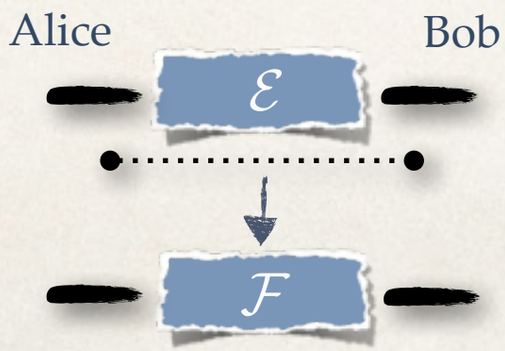


id: perfect classical channel
$\mathcal{E}$: quantum channel

Using entanglement, at what asymptotic rate can the classical id-channel simulate a quantum channel?

⇒ C_E as well! I.e. the asymptotic capacity of a quantum channel to simulate another quantum channel in the presence of free entanglement is given by:

$$C_E(\mathcal{E},\mathcal{F}) = \frac{C_E(\mathcal{E})}{C_E(\mathcal{F})}$$



Note: Maximally entangled states are not sufficient, embezzling states needed!

Embezzling States

- Introduced by Van Dam and Hayden [7]
- **Definition:** A pure, bipartite state of the form

$$|\mu(k)\rangle_{AB} = \frac{1}{\sqrt{G(k)}} \sum_{j=1}^{k} \frac{1}{\sqrt{j}} |jj\rangle_{AB}$$

where $G(k) = \sum_{j=1}^{k} \frac{1}{j}$, is called *embezzling state* of index k.

[7] Phys. Rev. A, Rapid Comm. 67:060302(R), 2003

Embezzling States

- Introduced by Van Dam and Hayden [7]
- **Definition:** A pure, bipartite state of the form

$$|\mu(k)\rangle_{AB} = \frac{1}{\sqrt{G(k)}} \sum_{j=1}^{k} \frac{1}{\sqrt{j}} |jj\rangle_{AB}$$

where $G(k) = \sum_{j=1}^{k} \frac{1}{j}$, is called *embezzling state* of index k.

- **Proposition:** Let $\epsilon > 0$ and let $|\varphi\rangle_{AB}$ be a pure bipartite state of Schmidt rank m. Then the transformation

$$|\mu(k)\rangle_{AB} \mapsto |\mu(k)\rangle_{AB} \otimes |\varphi\rangle_{AB}$$

can be accomplished with fidelity better than $(1 - \epsilon)$ for $k > m^{1/\epsilon}$ with local isometries at A and B.

- **Definition:** The *fidelity* between two density matrices ϱ and σ is defined as

$$F(\rho, \sigma) = (\mathrm{tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}))^2$$

and it is a notion of distance on the set of density matrices.

[7] Phys. Rev. A, Rapid Comm. 67:060302(R), 2003

Our Proof

- $\mathcal{E}_{A\to B}$ CPTP map to simulate, $\mathcal{E}_{A\to B}$: $S(\mathcal{H}_A) \to S(\mathcal{H}_B)$
  $\rho_A \mapsto \mathcal{E}_{A\to B}(\rho_A)$

- **Stinespring Dilation:**

$$\mathcal{E}_{A\to B}(\rho_A) = \mathrm{tr}_{A'}(U_{A\to BA'}\rho_A U_{A\to BA'}) =: \mathrm{tr}_{A'}(\sigma_{BA'})$$

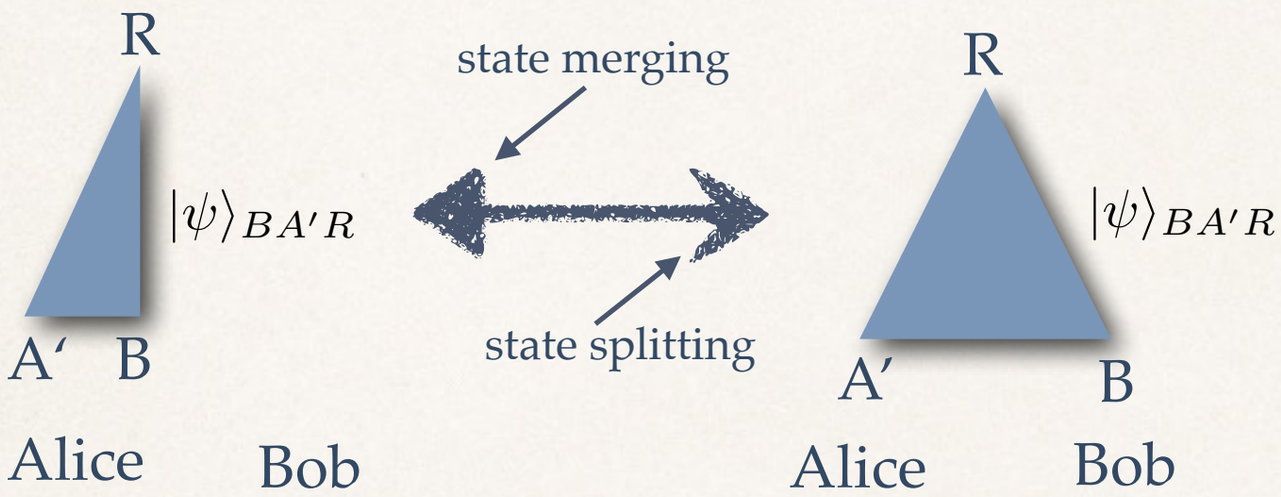
for some isometry $U_{A\to BA'} : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_{A'}$, with $\dim(\mathcal{H}_{A'}) \leq \dim(\mathcal{H}_A) \cdot \dim(\mathcal{H}_B)$.

- **Key Idea:**
  (i) Local simulation of $\mathcal{E}_{A\to B}$ at Alice's side using Stinespring Dilation
  $\Rightarrow \sigma_{BA'}$ at Alice's side.
  (ii) Send part B of $\sigma_{BA'}$ to Bob with classical channel and entanglement
  $\Rightarrow$ Bob has $\sigma_B = \mathcal{E}_{A\to B}(\rho_A)$!

Quantum State Merging/State Splitting

R: reference system

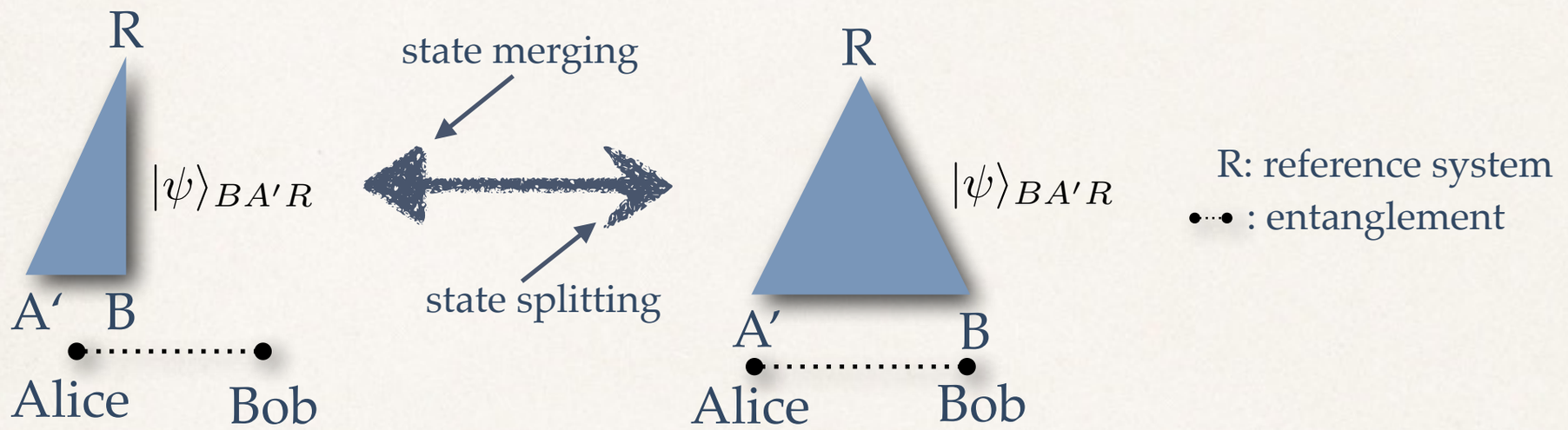
Quantum State Merging/State Splitting



R: reference system

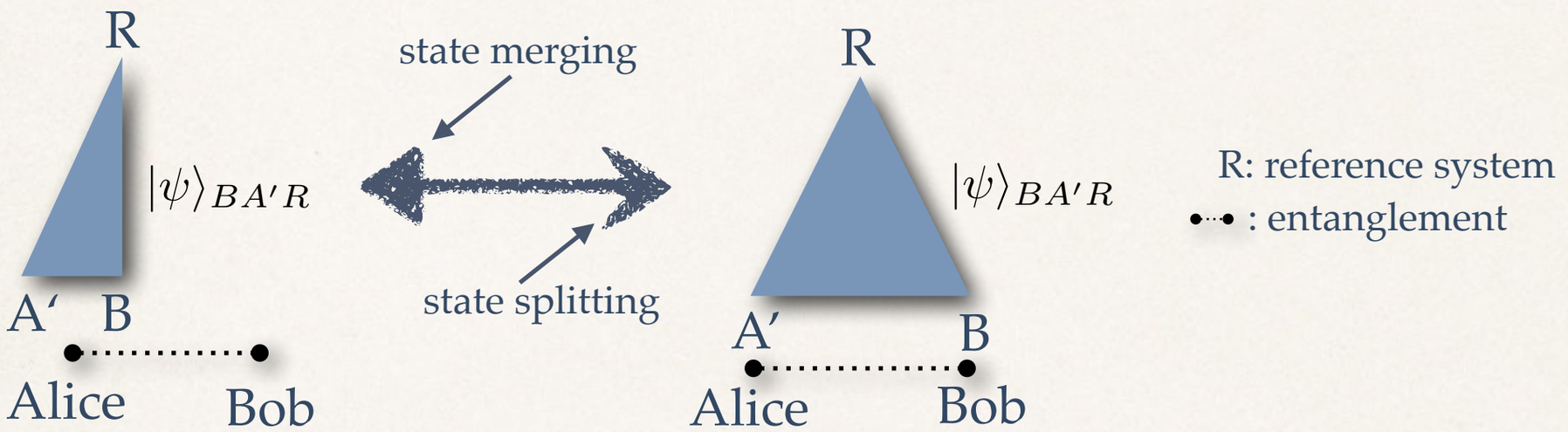
- How much of a given resource is needed to do this?

Quantum State Merging/State Splitting



- How much of a given resource is needed to do this?
- Our case: $\sigma_{BA'} \to \sigma_{BA'R} = |\psi\rangle\langle\psi|_{BA'R}$ purification, free entanglement, classical communication to quantify.

Quantum State Merging/State Splitting

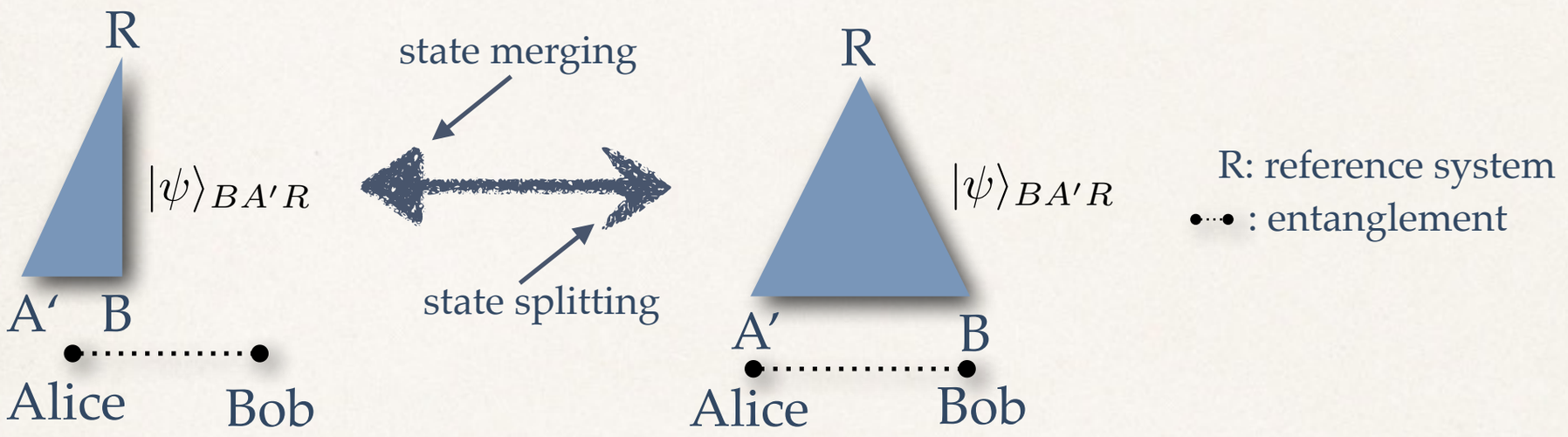


- How much of a given resource is needed to do this?
- Our case: $\sigma_{BA'} \rightarrow \sigma_{BA'R} = |\psi\rangle\langle\psi|_{BA'R}$ purification, free entanglement, classical communication to quantify.
- Horodecki et al. [2], $|\psi^{\otimes n}\rangle_{BA'R}$ with classical communication cost c_n :

$$c = \lim_{n\to\infty} \frac{1}{n} c_n = H(\sigma_B) + H(\sigma_R) - H(\sigma_{BR}) = I(B:R)_\sigma$$

[2] Nature 436:673-676, 2005

Quantum State Merging/State Splitting



- How much of a given resource is needed to do this?
- Our case: $\sigma_{BA'} \rightarrow \sigma_{BA'R} = |\psi\rangle\langle\psi|_{BA'R}$ purification, free entanglement, classical communication to quantify.
- Horodecki et al. [2], $|\psi^{\otimes n}\rangle_{BA'R}$ with classical communication cost c_n :

$$c = \lim_{n\to\infty} \frac{1}{n} c_n = H(\sigma_B) + H(\sigma_R) - H(\sigma_{BR}) = I(B:R)_\sigma$$

- One-shot version, $|\psi\rangle_{BA'R}$ with classical communication cost c_ϵ for an error ϵ :

$$c_\epsilon \cong I^\epsilon_{\max}(B:R)_\sigma$$

[2] Nature 436:673-676, 2005

Back to the Proof

- CPTP map $\mathcal{E}_{A\to B}^{\otimes n}(\rho_A^n) = \mathrm{tr}_{A'}(U_{A\to BA'}^n \rho_A^n U_{A\to BA'}^n) =: \mathrm{tr}_{A'}(\sigma_{BA'}^n)$ to simulate.

- Local simulation of $U_{A\to BA'}^n$ and state splitting of $\sigma_{BA'}^n$ gives ε-approximation $\mathcal{F}_{A\to B}^{n,\epsilon}$ of $\mathcal{E}_{A\to B}^{\otimes n}$ for a class. comm. cost $I_{\max}^{\epsilon}(B:R)_{\sigma^n}$.

Back to the Proof

- CPTP map $\mathcal{E}_{A\to B}^{\otimes n}(\rho_A^n) = \mathrm{tr}_{A'}(U_{A\to BA'}^n \rho_A^n U_{A\to BA'}^n) =: \mathrm{tr}_{A'}(\sigma_{BA'}^n)$ to simulate.

- Local simulation of $U_{A\to BA'}^n$ and state splitting of $\sigma_{BA'}^n$ gives ε-approximation $\mathcal{F}_{A\to B}^{n,\epsilon}$ of $\mathcal{E}_{A\to B}^{\otimes n}$ for a class. comm. cost $I_{\max}^{\epsilon}(B:R)_{\sigma^n}$.

- **Definition:** Let $\mathcal{E}$ be a quantum operation. The *diamond norm* [8] of $\mathcal{E}$ is defined as

$$\|\mathcal{E}\|_\diamond = \sup_{k\in\mathbb{N}} \sup_{\|\sigma\|_1\leq 1} \|(\mathcal{E}\otimes \mathrm{id}_k)(\sigma)\|_1$$

$$\|\sigma\|_1 = \mathrm{tr}(\sqrt{\sigma^\dagger\sigma})\,.$$

The induced metric is a notion of distance for quantum operations.

[8] Kitaev, Russian Math. Surveys 52:1191, 1997

Back to the Proof

- CPTP map $\mathcal{E}_{A\to B}^{\otimes n}(\rho_A^n) = \mathrm{tr}_{A'}(U_{A\to BA'}^n \rho_A^n U_{A\to BA'}^n) =: \mathrm{tr}_{A'}(\sigma_{BA'}^n)$ to simulate.

- Local simulation of $U_{A\to BA'}^n$ and state splitting of $\sigma_{BA'}^n$ gives ε-approximation $\mathcal{F}_{A\to B}^{n,\epsilon}$ of $\mathcal{E}_{A\to B}^{\otimes n}$ for a class. comm. cost $I_{\max}^{\epsilon}(B:R)_{\sigma^n}$.

- **Definition:** Let $\mathcal{E}$ be a quantum operation. The *diamond norm* [8] of $\mathcal{E}$ is defined as

$$\|\mathcal{E}\|_\diamond = \sup_{k\in\mathbb{N}} \sup_{\|\sigma\|_1\leq 1} \|(\mathcal{E}\otimes \mathrm{id}_k)(\sigma)\|_1$$

$$\|\sigma\|_1 = \mathrm{tr}(\sqrt{\sigma^\dagger\sigma})\,.$$

  The induced metric is a notion of distance for quantum operations.

- To show: $\lim_{\epsilon\to 0}\lim_{n\to\infty} \|\mathcal{E}_{A\to B}^{\otimes n} - \mathcal{F}_{A\to B}^{n,\epsilon}\|_\diamond = 0\,,\ \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{n} I_{\max}^{\epsilon}(B:R)_{\sigma^n} = C_E\,.$

[8] Kitaev, Russian Math. Surveys 52:1191, 1997

The Post-Selection Technique

- Christandl et al. [4]: Let $\mathcal{E}_{A^n}$ and $\mathcal{F}_{A^n}$ be quantum operations that act permutation-covariant on a n-partite system $\mathcal{H}_{A^n} = \mathcal{H}_A^{\otimes n}$. Then

$$\|\mathcal{E}_{A^n} - \mathcal{F}_{A^n}\|_\diamond \leq \mathrm{poly}(n)\|((\mathcal{E}_{A^n} - \mathcal{F}_{A^n}) \otimes \mathrm{id}_{R^n R'})(\zeta_{A^n R^n R'})\|_1$$

where $\zeta_{A^n R^n R'}$ is a purification of the (de Finetti type) state

$$\zeta_{A^n R^n} = \int \omega_{AR}^{\otimes n} d(\omega_{AR})$$

with ω_{AR} a pure state on $\mathcal{H}_A \otimes \mathcal{H}_R, \mathcal{H}_R \cong \mathcal{H}_A, \mathcal{H}_{R^n} = \mathcal{H}_R^{\otimes n}$ and $d(.)$ the measure on the normalized pure states on $\mathcal{H}_A \otimes \mathcal{H}_R$ induced by the Haar measure on the unitary group acting on $\mathcal{H}_A \otimes \mathcal{H}_R$, normalized to $\int d(.) = 1$.

[4] Phys. Rev. Lett. 102:020504, 2009

The Post-Selection Technique

- Christandl et al. [4]: Let $\mathcal{E}_{A^n}$ and $\mathcal{F}_{A^n}$ be quantum operations that act permutation-covariant on a n-partite system $\mathcal{H}_{A^n} = \mathcal{H}_A^{\otimes n}$. Then

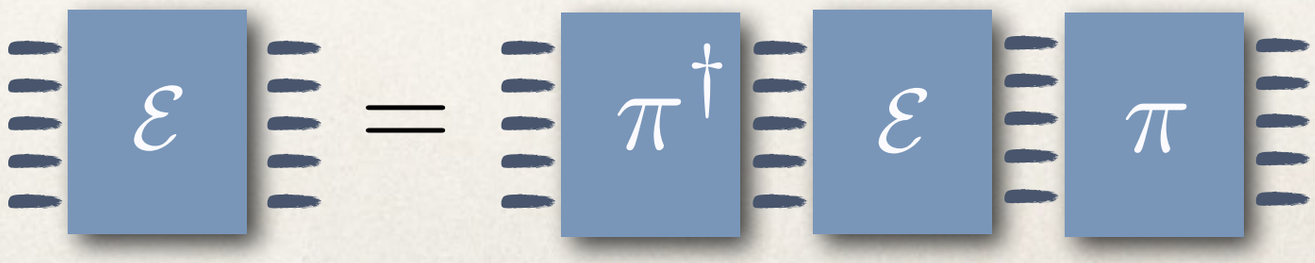
$$\|\mathcal{E}_{A^n} - \mathcal{F}_{A^n}\|_\diamond \leq \mathrm{poly}(n)\|((\mathcal{E}_{A^n} - \mathcal{F}_{A^n}) \otimes \mathrm{id}_{R^n R'})(\zeta_{A^n R^n R'})\|_1$$

where $\zeta_{A^n R^n R'}$ is a purification of the (de Finetti type) state

$$\zeta_{A^n R^n} = \int \omega_{AR}^{\otimes n} d(\omega_{AR})$$

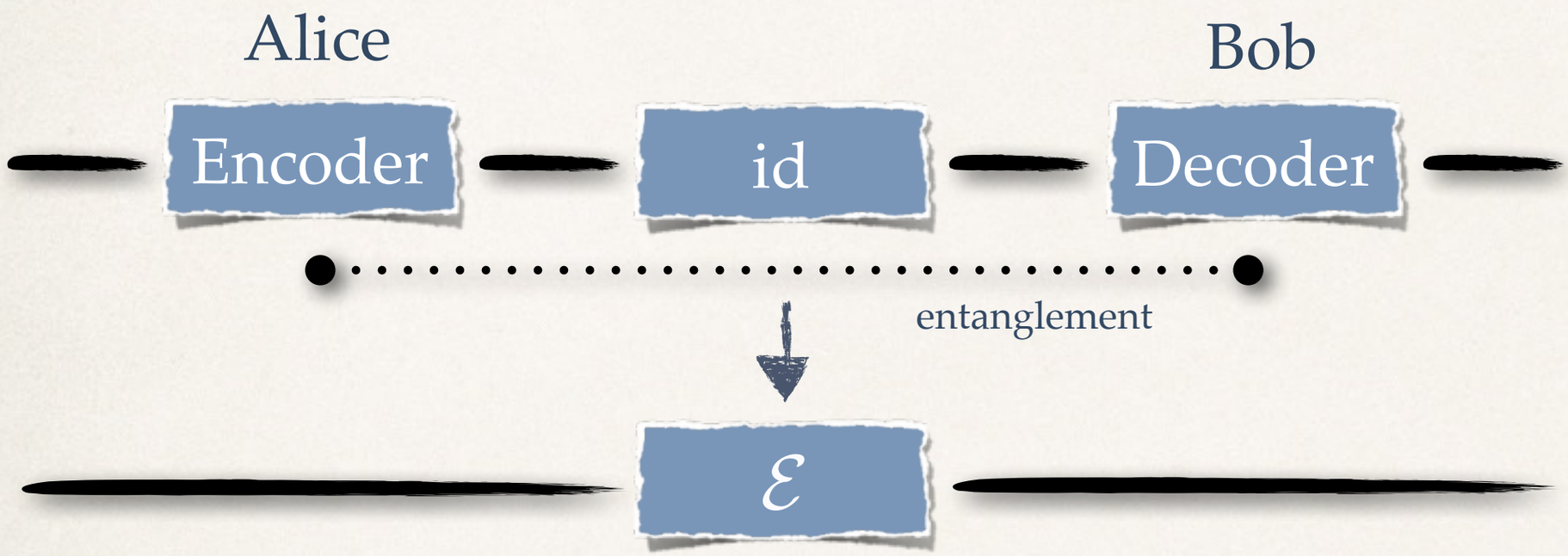
with ω_{AR} a pure state on $\mathcal{H}_A \otimes \mathcal{H}_R, \mathcal{H}_R \cong \mathcal{H}_A, \mathcal{H}_{R^n} = \mathcal{H}_R^{\otimes n}$ and $d(.)$ the measure on the normalized pure states on $\mathcal{H}_A \otimes \mathcal{H}_R$ induced by the Haar measure on the unitary group acting on $\mathcal{H}_A \otimes \mathcal{H}_R$, normalized to $\int d(.) = 1$.

- Permutation covariant:



[4] Phys. Rev. Lett. 102:020504, 2009

Conclusions



id: perfect classical channel
$\mathcal{E}$: quantum channel

Any quantum channel can be simulated by an unlimited amount of shared entanglement and an amount of classical communication equal to the channel's entanglement assisted classical capacity.

- Stinespring Dilation: $\mathcal{E}_{A\to B}^{\otimes n}(\rho_A^n) = \mathrm{tr}_{A'}(U_{A\to BA'}^n \rho_A^n U_{A\to BA'}^n) =: \mathrm{tr}_{A'}(\sigma_{BA'}^n)$

- Local simulation of $U_{A\to BA'}^n$ and (optimal) one-shot State Splitting of $\sigma_{BA'}^n$ gives ε-approximation $\mathcal{F}_{A\to B}^{n,\epsilon}$ of $\mathcal{E}_{A\to B}^{\otimes n}$. Using Post-Selection Technique everything works!