Identifying the Information Gain of a Quantum Measurement

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***Abstract*—We show that quantum-to-classical channels, i.e., quantum measurements, can be asymptotically simulated by an amount of classical communication equal to the quantum mutual information of the measurement, if sufficient shared randomness is available. This result generalizes Winter's measurement compression theorem for fixed independent and identically distributed inputs [Winter, CMP 244 (157), 2004] to arbitrary inputs, and more importantly, it identifies the quantum mutual information of a measurement as the information gained by performing it, independent of the input state on which it is performed. Our result is a generalization of the classical reverse Shannon theorem to quantum-to-classical channels. In this sense, it can be seen as a quantum reverse Shannon theorem for quantum-to-classical channels, but with the entanglement assistance and quantum communication replaced by shared randomness and classical communication, respectively. Our proof is based on quantum-proof randomness extractors and the post-selection technique for quantum channels [Christandl *et al.*, PRL 102 (020504), 2009].**

I. Introduction

Measurement is an integral part of quantum theory. It is the means by which we gather information about a quantum system. Although the classical notion of a measurement is rather straightforward, the quantum notion of measurement has been the subject of much thought and debate [1]. One interpretation is that the act of measurement on a quantum system causes it to abruptly jump or "collapse" into one of several possible states with some probability, an evolution seemingly different from the smooth, unitary transitions resulting from Schrödinger's wave equation. Some have advocated for a measurement postulate in quantum theory [14], while others have advocated that our understanding of quantum measurement should follow from other postulates [43].

In spite of the aforementioned difficulties in understanding and interpreting quantum measurement, there is a precise question that one can formulate concerning it:

> *How much information is gained by performing a given quantum measurement?*

This question has a rather long history, which to our knowledge begins with the work of Groenewold [17]. In 1971, Groenewold argued on intuitive grounds for the following "entropy reduction" to quantify the information gained by performing a quantum measurement:

$$H(\rho) - \sum_x p_x H(\rho_x) \ , \tag{1}$$

where ρ is the initial state before the measurement occurs, $\{p_x, \rho_x\}$ is the post-measurement ensemble induced by the measurement, and $H(\sigma) \equiv -\text{tr}[\sigma \log \sigma]$ is the von Neumann entropy of a state σ. The intuition behind this measure is that it quantifies the reduction in uncertainty after performing a quantum measurement on a quantum system in state ρ, and its form is certainly reminiscent of a Holevo-like quantity [19], although the classical data in the above Groenewold quantity appears at the output of the process rather than at the input as in the case of the Holevo quantity. Groenewold left open the question of whether this quantity is non-negative for all measurements, and Lindblad proved that non-negativity holds whenever the measurement is of the von Neumann-Lüders kind (projecting onto an eigenspace of an observable) [29]. Ozawa then settled the matter by proving that the above quantity is non-negative if and only if the post-measurement states are of the form

$$\rho_x = M_x \rho M_x^\dagger \,/\, \text{tr}[M_x^\dagger M_x \rho] \ , \tag{2}$$

for some operators $\{M_x\}$ such that $\sum_x M_x^\dagger M_x = \mathbb{1}$ [32].

The fact that the quantity in (1) can become negative for some quantum measurements excludes it from being a generally appealing measure of information gain. To remedy this situation, Buscemi *et al*. later advocated for the following measure to characterize the information gain of a quantum measurement when acting upon a particular state ρ [8], [30], [35], [40]:

$$I(X:R)_\omega \ , \tag{3}$$

where $I(X:R)_\omega \equiv H(X)_\omega + H(R)_\omega - H(XR)_\omega$ is the quantum mutual information of the state:

$$\omega_{XR} \equiv \sum_x |x\rangle\langle x|_X \otimes \text{tr}_A\{(\mathcal{M}_x \otimes \mathcal{I}_R)(|\rho\rangle\langle\rho|_{AR})\} \ . \tag{4}$$

The register X is a classical register containing the outcome of the measurement, $\mathcal{M} \equiv \{\mathcal{M}_x\}$ is a collection of completely positive, trace non-increasing maps characterizing the measurement (for which the sum map $\sum_x \mathcal{M}_x$ is trace preserving), $\mathcal{I}$ is the identity map, and $\rho_{AR} = |\rho\rangle\langle\rho|_{AR}$ is a purification of the initial state ρ_A on system A to a purifying system R.

The advantages of the measure of information gain in (3) are as follows:

- It is non-negative.

- It reduces to Groenewold's quantity in (1) for the special case of measurements of the form in (2) [8].
- It characterizes the trade-off between information and disturbance in quantum measurements [8].
- It has an operational interpretation in Winter's measurement compression protocol as the optimal rate at which a measurement gathers information [42].

This last advantage is the most compelling one from the perspective of quantum information theory—one cannot really justify a measure as an information measure unless it corresponds to a meaningful information processing task. Indeed, when reading the first few paragraphs of Groenewold's paper [17], it becomes evident that his original motivation was information theoretic in nature, and with this in mind, Winter's measure in (3) is clearly the one Groenewold was seeking after all.

In spite of the above arguments in favor of the information measure in (3) as a measure of information gain, it is still lacking in one aspect: it is dependent on the state on which the quantum measurement $\mathcal{M}$ acts in addition to the measurement itself. A final requirement that one should impose for a measure of information gain by a measurement is that it should depend only on the measurement itself. A simple way to remedy this problem is to maximize the quantity in (3) over all possible input states, leading to the following characterization of information gain:

$$I(\mathcal{M}) \equiv \max_{\rho_{AR}} I(X:R)_\omega \ , \tag{5}$$

for ω_{XR} as in (4). The quantity above has already been identified and studied by previous authors as an important information quantity, being labeled as the purification capacity of a measurement [25], [26] or the information capacity of a quantum observable [22]. The above quantity also admits an operational interpretation as the entanglement-assisted capacity of a quantum measurement for transmitting classical information [3], [21], [22], though it is our opinion that this particular operational interpretation is not sufficiently compelling such that we should associate the measure in (5) with the notion of information gain. The main aim of this paper is to address this issue by providing a compelling operational interpretation of the measure in (5).

II. Results

Our main contribution is to show that $I(\mathcal{M})$ is the optimal rate at which a measurement gains information when many identical instances of it act on an arbitrary input state (see Ref. [7] for a full exposition of our results). In our opinion, this new result establishes (5) as the information-theoretic measure of information gain of a quantum measurement. In more detail, let A denote the input Hilbert space for a given measurement $\mathcal{M}$. We suppose that a third party prepares an arbitrary quantum state on a Hilbert space $A^{\otimes n}$, which is equivalent to n identical copies of the original Hilbert space A, where n is a large positive number. A sender and receiver can then exploit some amount of shared random bits and classical communication to simulate the action of n instances of the measurement $\mathcal{M}$ (denoted by $\mathcal{M}^{\otimes n}$) on the chosen input state, in such a way that it becomes physically impossible for the third party, to whom the receiver passes along the measurement outcomes, to distinguish between the simulation and the ideal measurement $\mathcal{M}^{\otimes n}$ as n becomes large (the third party can even keep the purifying system of a purification of the chosen input state in order to help with the distinguishing task). By design, the information gained by the measurement is that relayed by the classical communication. Following [42], we call this task universal measurement compression. We prove that the optimal rate of classical communication is equal to $I(\mathcal{M})$, if sufficient shared randomness is available.

The information-theoretic task outlined above is also known as channel simulation, and it has been well studied for the case of fully classical channels (with classical inputs and classical outputs) [3], [10], [11] and fully quantum channels (with quantum inputs and quantum outputs) [2], [3], [6]. The in-between case of channels with quantum inputs and classical outputs (i.e., measurements) has been studied as well [42] (see also [40]), but as mentioned above, the problem of simulating many instances of a quantum measurement on an arbitrary input state has not been studied before this paper. Beyond its intrinsic interest as an information-processing task, channel simulation has two known concrete applications: in establishing a strong converse rate for a channel coding task [2]–[5] and in rate distortion coding (lossy data compression) [12], [13], [39], [41].

The precise information-theoretic formulation of our results is as follows. We characterize the optimal rate region consisting of the rates of shared randomness and classical communication that are both necessary and sufficient for the existence of a measurement simulation, whenever both the sender and receiver are required to obtain the measurement outcomes (this is known as a feedback simulation since the sender also obtains the measurement outcomes).

Theorem 1 (Feedback Universal Measurement Compression)**.** Let $\mathcal{M} = \{\mathcal{M}_x\}$ be a quantum-to-classical channel. Then there exist asymptotic feedback measurement compressions for $\mathcal{M}$ if and only if the classical communication rate C and shared randomness rate S lie in the following rate region,[1]

$$C \geq \max_{\rho} I(X:R)_\omega \tag{6}$$

$$C + S \geq \max_{\rho} H(X)_\omega \ , \tag{7}$$

where $\rho_{AR} = |\rho\rangle\langle\rho|_{AR}$ is a purification of the input state ρ_A, and ω_{XR} as in (4). Or equivalently, for a given shared randomness rate S, the optimal rate of classical communication is equal to

$$C(S) = \max\left\{ \max_{\rho} I(X:R)_\omega,\ \max_{\rho} H(X)_\omega - S \right\} \ . \tag{8}$$

[1]Note that the two maxima in (6) and (7) can be achieved for different states.

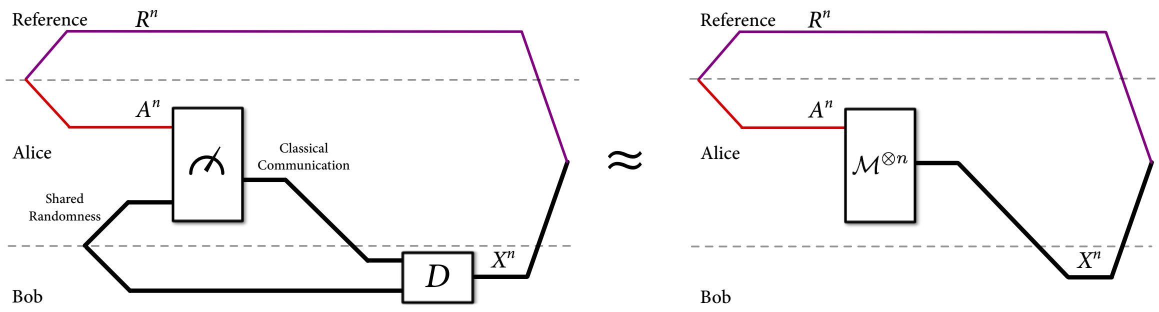

Fig. 1. Simulation (left) of the measurement $\mathcal{M}^{\otimes n}$ (right). In the simulation, Alice uses shared randomness to perform a new measurement, whose result she communicates to Bob, such that Bob can recover the actual measurement output X^n using the message and the shared randomness. If the simulation scheme works for any input, we can associate the amount of communication with the information gained by the measurement.

In particular, when sufficient shared randomness is available, the rate of classical communication is given by

$$C(\infty) = \max_{\rho} I(X:R)_{\omega} \ . \tag{9}$$

We also characterize the optimal rate region of shared randomness and classical communication for a non-feedback simulation, in which the sender is not required to obtain the measurement outcomes.

Theorem 2 (Non-Feedback Universal Measurement Compression)**.** Let $\mathcal{M} = \{\mathcal{M}_x\}$ be a quantum-to-classical channel. Then there exist asymptotic non-feedback measurement compressions for $\mathcal{M}$ if and only if the classical communication rate C and shared randomness rate S lie in the rate region given by the union of the following regions,

$$C \geq \max_{\rho} I(W:R)_{\beta} \tag{10}$$

$$C + S \geq \max_{\rho} I(W:XR)_{\beta} \ , \tag{11}$$

where the state β_{WXR} has the form

$$\beta_{WXR} = \sum_{w,x} q_{x|w} \cdot |w\rangle\langle w|_W \otimes |x\rangle\langle x|_X \otimes \mathrm{tr}_A\big[(\mathcal{N}_w \otimes \mathcal{I})(\rho_{AR})\big] \ , \tag{12}$$

$\rho_{AR} = |\rho\rangle\langle\rho|_{AR}$ is a purification of the input state ρ_A, and the union is with respect to all decompositions of the measurement $\mathcal{M}$ in terms of internal measurements $\mathcal{N} = \{\mathcal{N}_w\}$ and conditional post-processing distributions $q_{x|w}$. That is, for all states σ, it should hold that

$$\sum_x \mathcal{M}_x(\sigma)|x\rangle\langle x| = \sum_{x,w} q_{x|w}\, \mathcal{N}_w(\sigma)|x\rangle\langle x|. \tag{13}$$

Or equivalently, for a given shared randomness rate S, the optimal rate of classical communication is equal to

$$C(S) = \min_{\mathcal{N}:\sum_w q_{x|w}\mathcal{N}_w = \mathcal{M}_x} \max\Big\{ \max_{\rho} I(W:R)_{\beta}, \max_{\rho} I(W:XR)_{\beta} - S \Big\} \ . \tag{14}$$

By the data processing inequality for the mutual information, it holds that $I(W:R)_{\beta} \geq I(X:R)_{\omega}$, and hence, the classical communication cost can only increase compared to a feedback simulation (Theorem 1). However, if the savings in common randomness consumption are larger than the increase in classical communication cost, then there is an advantage to performing a non-feedback simulation. It follows from the considerations in [31], [40] that the rate trade-offs (8) and (14) become identical if and only if the elements of the measurement to simulate are all rank-one operators.

By applying our simulation results (Theorem 1 and Theorem 2) to classical channels, we also see that the classical reverse Shannon theorem [2], [3] is a strict specialization of universal measurement compression.

III. Proof Ideas

Our proof technique exploits ideas from the approach in [6] for proving the fully quantum reverse Shannon theorem. In fact, one can think of our approach here as a classicalized or dephased version of that approach. In particular, the proof is also based on one-shot information theory and uses the smooth entropy formalism [33], [37]. In addition, we make use of the idea of classically coherent states. We say that a pure state $\rho_{X_A X_B R} = |\rho\rangle\langle\rho|_{X_A X_B R}$ is classically coherent with respect to systems $X_A X_B$ if there is an orthonormal basis $\{|x\rangle\}$ such that $|\rho\rangle$ can be written in the form

$$|\rho\rangle_{X_A X_B R} = \sum_x \sqrt{p_x}\, |xx\rangle_{X_A X_B} \otimes |\rho_x\rangle_R \ , \tag{15}$$

for some probability distribution $\{p_x\}$, and pure states $\rho_x = |\rho_x\rangle\langle\rho_x|_R$. Harrow realized the importance of classically coherent states for quantum communication tasks [18], while [16] recently exploited this notion in devising a decoupling approach to the Holevo-Schumacher-Westmoreland coding theorem [20], [34] that is useful for our purposes here.

We begin by establishing a protocol known as classically coherent state merging, which is a variation of the well-known state merging protocol [23], [24] specialized to classically coherent states. For the analysis we require quantum-proof min-entropy extractors [36]. We then show how time-reversing this protocol and exchanging the roles of Alice and Bob leads to a protocol known as classically coherent state splitting. It suffices for our purposes for this protocol to use shared randomness and classical communication rather than entanglement and quantum communication, respectively. Generalizing this last protocol then leads to a one-shot channel simulation which is essentially optimal when acting on a single copy of a known state. To state the cost function of this protocol we make use of the smooth entropy formalism [33], [37].

Proposition 3 (One-Shot Measurement Compression)**.** Let $\mathcal{M} = \{\mathcal{M}_x\}$ be a quantum-to-classical channel, ρ and input quantum state, and $\varepsilon > 0$. Then there exist an ε-error one-shot measurement compression for $\mathcal{M}$ with input ρ if for the classical communication cost c and the shared randomness cost s,

$$c \gtrsim I^{\varepsilon}_{\max}(X:R)_{\omega} + O(\log(1/\varepsilon)) \tag{16}$$

$$c + s \gtrsim H^{\varepsilon}_{\max}(X)_{\omega} \ , \tag{17}$$

with ω_{XR} as in (4), $I^{\varepsilon}_{\max}(X : R)_{\omega}$ the smooth max-information [6], and $H^{\varepsilon}_{\max}(X)_{\omega}$ the smooth max-entropy [28].

The max-information that B has about A is defined as

$$I_{\max}(A:B)_{\rho} = \min_{\sigma_B} \min \left\{ \lambda : 2^{\lambda} \cdot \rho_A \otimes \sigma_B \geq \rho_{AB} \right\} \ , \tag{18}$$

and the max-entropy of A is defined as

$$H_{\max}(A)_{\rho} = 2 \log \operatorname{tr} \left[\rho_A^{1/2} \right] \ . \tag{19}$$

The smooth measures are then defined by extremizing the non-smooth measures over a set of nearby states $\mathcal{B}^{\varepsilon}(\cdot)$, that is,

$$I^{\varepsilon}_{\max}(A:B)_{\rho} = \min_{\bar{\rho} \in \mathcal{B}^{\varepsilon}(\rho)} I_{\max}(A:B)_{\bar{\rho}} \tag{20}$$

$$H^{\varepsilon}_{\max}(A)_{\rho} = \min_{\bar{\rho} \in \mathcal{B}^{\varepsilon}(\rho)} H_{\max}(A)_{\bar{\rho}} \ . \tag{21}$$

Finally, we exploit the post-selection technique for quantum channels [9], the asymptotic equipartition property for the smooth max-information [6] and the smooth max-entropy [38], and the aforementioned state splitting protocol to show that it suffices to simulate many instances of a measurement on a purification of a particular de Finetti quantum input state in order to guarantee that the simulation is asymptotically perfect when acting on an arbitrary quantum state. For the non-feedback case we additionally need the idea of randomness recycling [2], and use a special smoothing based on typical projectors.

We note that the case of a fixed IID source from Winter's original paper on measurement compression [42] also follows easily from our analysis. We can simply apply the one-shot protocol from Proposition 3 to the case of a fixed IID source and then invoke the asymptotic equipartition property. In this way, we provide a more modern proof of this special case that avoids the use of typical projectors and the operator Chernoff bound. Finally, in Winter's paper additional arguments were required to establish that a POVM (positive operator-valued measure) compression protocol can function as an instrument compression protocol, where for an instrument compression protocol, Alice and Bob receive the classical outcomes of the measurement while Alice obtains the post-measurement states (see Section V of [42]). We note that our protocol here already functions as an instrument compression protocol due to our use of the classical state splitting protocol as a coding primitive.

IV. Conclusion

We have justified the information-theoretic measure in (5) as quantifying the information gain of a quantum measurement, by providing an operational interpretation in terms of a protocol for universal measurement compression. The main tools used to prove this result are the post-selection technique for quantum channels and a novel classical state splitting protocol based on quantum-proof extractors.

There are a number of open questions to consider going forward from here. Given that there are applications of "information gain" or "entropy reduction" in thermodynamics [27] and quantum feedback control [15], it would be interesting to explore whether the quantity in (5) has some application in these domains. Also, Buscemi *et al.* showed that the static measure of information gain in (3) plays a role in quantifying the trade-off between information extraction and disturbance [8], and it would be interesting to determine if there is a role in this setting for the information quantity in (5).

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