Entropy Inequalities

Mario Berta

Institute for Quantum Information and Matter
California Institute of Technology

Based on joint work with Marius Lemm, Kaushik Seshadreesan, Marco Tomamichel, Mark Wilde

Caltech

Entropy I

- Shannon entropy of random variable $(\mathcal{X}, \{p_x\}_{x\in\mathcal{X}})$:

$$H(X) := -\sum_{x\in\mathcal{X}} p_x \log p_x$$

—> "measure of uncertainty". Ex: $H(X)_\delta = 0$, $H(X)_{\mathrm{unif}} = \log|\mathcal{X}|$

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- Von Neumann entropy of quantum state ρ_A:

$$H(A)_\rho := -\mathrm{tr}\left[\rho_A \log \rho_A\right]$$

$\left(\rho_A \in \mathrm{Lin}(\mathcal{H}_A) \text{ with } \rho_A \geq 0,\ \mathrm{tr}\left[\rho_A\right] = 1\right)$

[von Neumann (32)]

Ex: $H(A)_\psi = 0$, $H(A)_{\frac{1}{|A|}} = \log|A|$

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- I would like to study the mathematical properties of this quantity.

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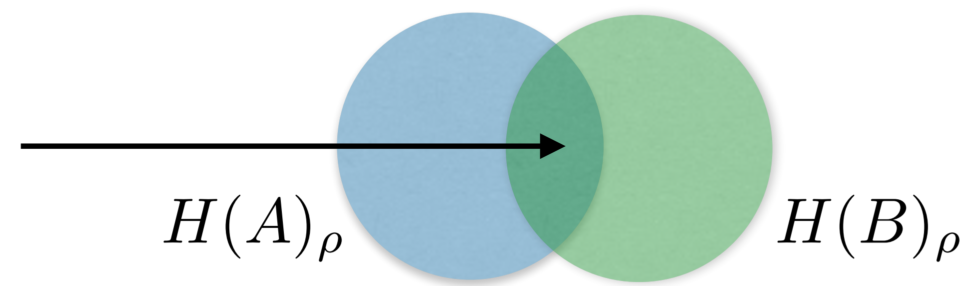
$$I(A:B)_\rho := H(A)_\rho + H(B)_\rho - H(AB)_\rho$$



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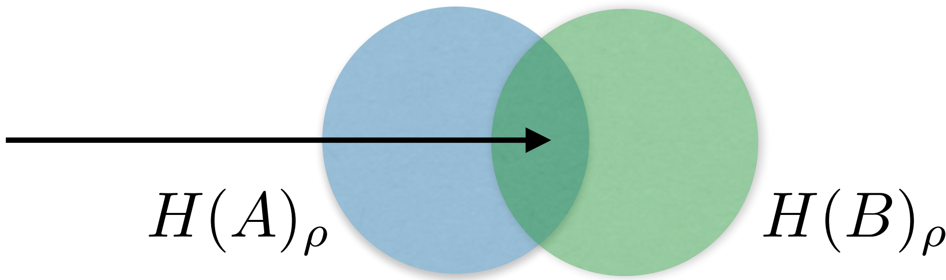

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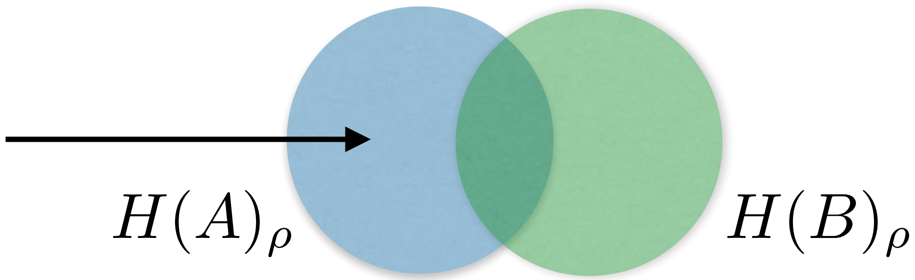

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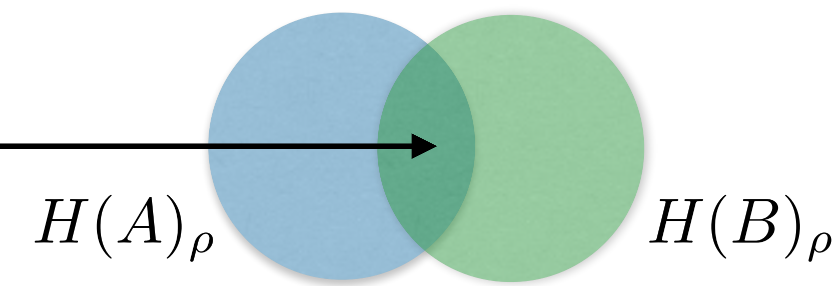

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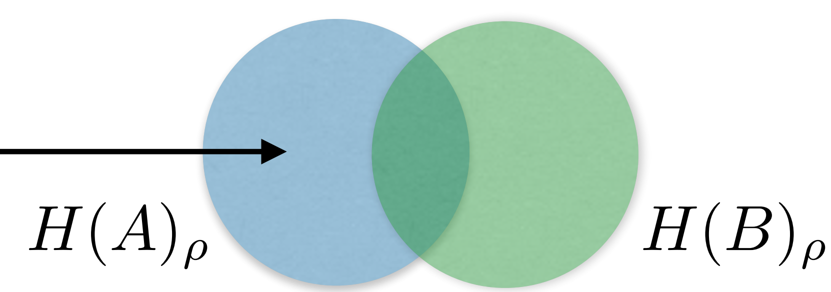

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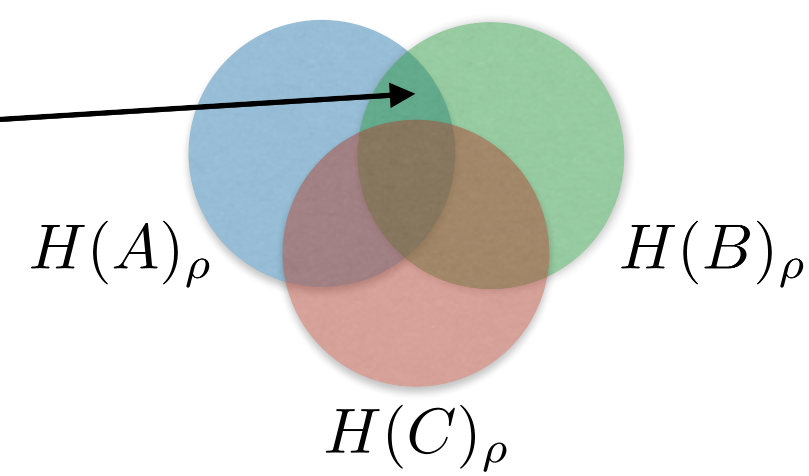

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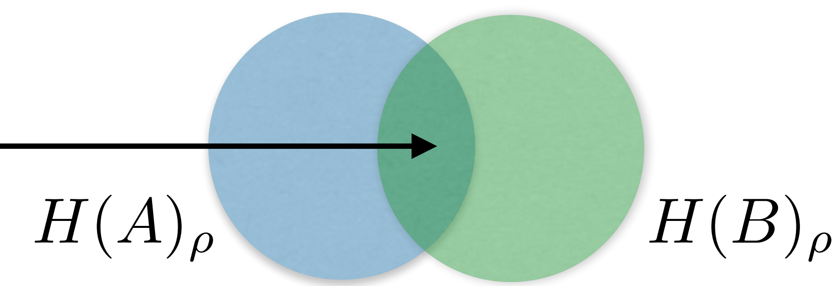

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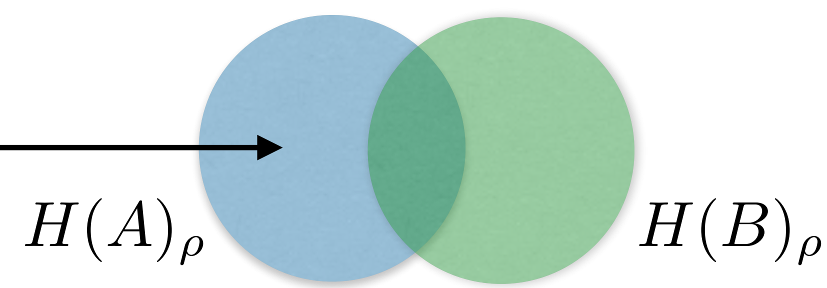

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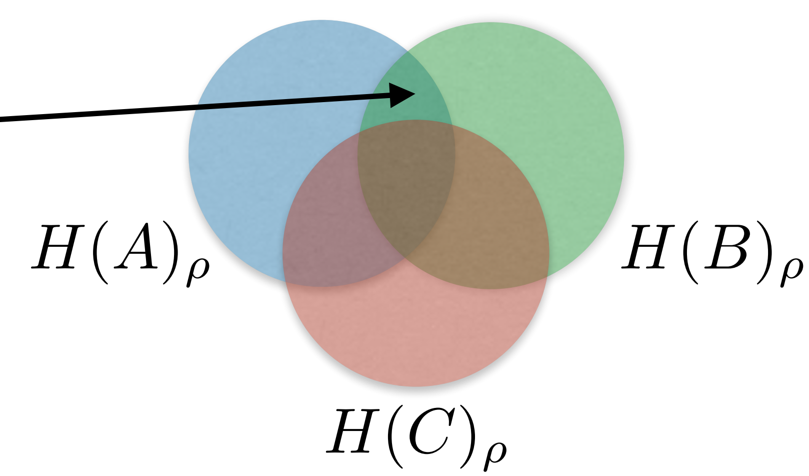

- … (other combinations, more parties, etc.)

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- Entropy - operational significance
- Entropy inequalities - laws of information theory
- Recent progress on refining these laws
- Extension: quantum relative entropy and its inequalities
- Conclusions

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- **Compression**: $A \overset{\text{Enc}}{\to} M \overset{\text{Dec}}{\to} A$

  —> asymptotic rate for compression of $\rho_A^{\otimes n}$ is

$$\lim_{n\to\infty} \frac{|M^n|}{n} = H(A)_\rho$$

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- **Communication** over channel: $M \overset{\text{Enc}}{\to} \mathcal{N}_{A\to B} \overset{\text{Dec}}{\to} M$

  —> asymptotic rate of transmission (entanglement assisted) for $\mathcal{N}_{A\to B}^{\otimes n}$ is

  $\lim_{n\to\infty} \frac{|M^n|}{n} = \max_\rho I(B:R)_{\mathcal{N}(\rho)}$ where $(\mathcal{N}_{A\to B} \otimes \mathcal{I}_R)(\rho_{AR})$ with ρ_{AR} pure

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- **Entanglement manipulation** (distillation, dilution, etc.)

- **Distributed compression**: quantum state merging, quantum state splitting, the mother protocol, quantum state redistribution etc.

- **...**

Operational Significance II

- Entropy, conditional entropy, mutual information, conditional mutual information etc. crucial (tool) for:
  - Quantum Shannon theory (cf. last slide)
  - Entanglement / correlation measures
  - Entropic uncertainty relations
  - Entanglement in quantum many body systems
  - Quantum error correction
  - Quantum statistical mechanics
  - Thermodynamics
  - Quantum communication complexity
  - Quantum de Finetti theorems
  - …

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- **Strong subadditivity of entropy (SSA)**:
  $$I(A:B|C)_\rho \geq 0$$
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[Lieb, Ruskai (73)]

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  —> generates all other inequalities,
  "all we know about entropy"!

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Entropy Inequalities II

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  [Ibinson, Linden, Winter (06)] [Li, Winter (12)]
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- What are the equality conditions of SSA?

$$I(A:B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = \left(\mathcal{I}_A \otimes \Lambda^{\text{Petz}}_{C\to BC}\right)(\rho_{AC})$$

recoverable states
[Petz (88)]

with $\Lambda^{\text{Petz}}_{C\to BC}(\cdot) := \rho_{BC}^{1/2}\rho_C^{-1/2}(\cdot)\rho_C^{-1/2}\rho_{BC}^{1/2}$

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- This characterises quantum states with conditional quantum mutual information equal to zero, but what about: $I(A:B|C)_\rho \approx 0 \Rightarrow \rho_{ABC} = ?$

—> maybe: $I(A:B|C)_\rho \geq f(\rho_{ABC}, \Lambda^{\text{Petz}}_{C\to BC}(\rho_{AC}))$?

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Refinements I

- One way of lower bounding entropy is via **quantum Renyi entropies**:

$$H_\alpha(A)_\rho := \frac{1}{1-\alpha} \log \mathrm{tr}\left[\rho_A^\alpha\right], \quad \alpha \geq 0$$ [Renyi (61)]

  - von Neumann entropy: $H_1(A)_\rho = H(A)_\rho$
  - monotone in Renyi parameter: $\alpha \geq \beta \quad \Rightarrow \quad H_\alpha(A)_\rho \geq H_\beta(A)_\rho$
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- Define **Renyi conditional quantum mutual information**:

$I_\alpha(A:B|C)_\rho = \ldots$ —> with: $I_1(A:B|C)_\rho = I(A:B|C)_\rho$

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- Monotone? $\alpha \geq \beta \quad \Rightarrow \quad I_\alpha(A:B|C)_\rho \geq I_\beta(A:B|C)_\rho$?

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- **Conjecture:** $$I(A:B|C)_\rho \geq -\log F(\rho_{ABC}, \Lambda^{\text{Petz}}_{C\to BC}(\rho_{AC})) \ ?$$

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$$F(\rho,\sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 \quad \|\rho\|_1 := \mathrm{tr}\left[\sqrt{\rho^\dagger \rho}\right]$$

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- This would give a characterisation of states with small conditional quantum mutual information:

$I(A:B|C)_\rho \approx 0 \Rightarrow \rho_{ABC} = ?$ —> we had: $F\left(\rho_{ABC}, \Lambda^{\text{Petz}}_{C\to BC}(\rho_{AC})\right) \approx 1$

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- However, we only have proofs for special cases (analytical evidence) and numerical evidence…

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- **Recent breakthrough:**

$$I(A:B|C)_\rho \geq -\log F(\rho_{ABC}, \mathcal{V}_{BC} \circ \Lambda^{\text{Petz}}_{C\to BC} \circ \mathcal{U}_C(\rho_{AC}))$$

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- Applications so far: understanding quantum correlations better

[Wilde, Seshadreesan (14)] [Wilde (14)] [Li, Winter (14)][Piani (15)]

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- Extension: quantum relative entropy and its inequalities
- Conclusions

Quantum Relative Entropy I

- Parent quantity: $$D(\rho\|\sigma) := \mathrm{tr}\left[\rho \log \rho\right] - \mathrm{tr}\left[\rho \log \sigma\right]$$ [Umegaki (62)]

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- Example: strong subadditivity of entropy (SSA)

$$\rho = \rho_{ABC},\ \sigma = 1_A \otimes \rho_{BC},\ \mathcal{N}(\cdot) = \mathrm{tr}_B[\cdot] \ \Rightarrow\ \mathcal{N}(\rho) = \rho_{AC},\ \mathcal{N}(\sigma) = 1_A \otimes \rho_C$$
$$0 \leq D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = -H(A|BC)_\rho + H(A|C)_\rho = I(A:B|C)_\rho$$

Quantum Relative Entropy II

- Can we improve MONO (in an operationally useful way)? [Li, Winter (12, 14)]

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq 0$$

vs.

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma) \geq f(\rho,\sigma,\mathcal{N})\ ?$$

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- What are the equality conditions for MONO?

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma) = 0 \ \Leftrightarrow \ \rho = \Lambda^{\mathrm{Petz}}_{\mathcal{N},\sigma}(\mathcal{N}(\rho)) \\ \sigma = \Lambda^{\mathrm{Petz}}_{\mathcal{N},\sigma}(\mathcal{N}(\sigma))$$

[Petz (88)]

with $\Lambda^{\mathrm{Petz}}_{\mathcal{N},\sigma}(\cdot) := \sigma^{1/2}\mathcal{N}^{\dagger}\left(\mathcal{N}(\sigma)^{-1/2}(\cdot)\mathcal{N}(\sigma)^{-1/2}\right)\sigma^{1/2}$

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—> this is a quantum operation, recovery map.

- If the relative entropy difference is zero we can undo noisy quantum operation!
- But we also need to understand the approximate case…

Quantum Relative Entropy III

- **Conjecture:**

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -\log F\left(\rho, \Lambda^{\mathrm{Petz}}_{\mathcal{N},\sigma}(\mathcal{N}(\rho))\right) \ ?$$

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—> however, we would like to know more about the unitaries…

Equivalence

- The following are **equivalent**:
  - Strong subadditivity of entropy (SSA)
    $$I(A:B|C)_\rho \geq 0$$
  - Monotonicity of relative entropy under quantum operations (MONO):
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- **Refinements** in terms of Petz recovery map are **equivalent as well**:

$$I(A:B|C)_\rho \geq -\log F(\rho_{ABC}, \Lambda^{\text{Petz}}_{C\to BC}(\rho_{AC})) \;\Leftrightarrow\; D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -\log F\left(\rho, \Lambda^{\text{Petz}}_{\mathcal{N},\sigma}(\mathcal{N}(\rho))\right)$$
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—> however, we do not know if they actually hold: either all of these refinements (in terms of the Petz recovery map) are true or all are wrong…

Outline

- Entropy - operational significance
- Entropy inequalities - laws of information theory
- Recent progress on refining these laws
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- Strong subadditivity of entropy (SSA):

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extended to

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- Petz recovery map form to (dis)prove, many potential applications