One-shot multiple access channel simulation

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***Abstract*—We consider the problem of simulating a two-sender multiple access channel (MAC) for fixed product inputs, where each sender transmits a message to the decoder over a rate-limited noiseless link based on its input and unlimited randomness shared with the decoder. As our main contribution, we characterize the one-shot communication cost region via almost-matching inner and outer bounds phrased in terms of the smooth max-information of the channel. The achievability relies on a rejection-sampling algorithm to simulate a quantization channel between each sender and decoder, and producing the final output based on the output of these intermediate channels. The converse follows via information-spectrum based arguments relating operational quantities to information measures. Our one-shot results recover the single-letter asymptotic rate region for MAC simulation with fixed, independent and identically distributed product inputs, that was obtained in [Kurri *et al.*, IEEE Transactions on Information Theory 68, 7575 (2022)]. We extend our result to quantum-to-classical channels with a separable decomposition [Atif *et al.*, IEEE Transactions on Information Theory 68, 1085 (2022)], for which we obtain a similar characterization.**

I. Introduction

The channel simulation problem deals with the task of quantifying the minimum amount of communication required to establish correlation remotely, as dictated by the input-output joint distribution of the channel to be simulated. The most basic point to point channel simulation setup consists of an encoder-decoder pair that with access to shared randomness and communication over a noiseless rate-limited link achieves the channel simulation task. More specifically, the encoder observes a random variable, say X with distribution q_X, and based on the shared randomness, sends a message to the decoder. Based on this message and shared randomness, the decoder outputs a random variable Y. The aim of the protocol is to ensure that the trace distance between the joint distribution (X, Y) and the joint distribution induced by passing the source X through a discrete memory less channel $q(Y|X)$ is as small as possible. The channel simulation task is closely related to the task of creating a desired joint distribution between two distributed parties, also known as strong coordination [1].

Here, we consider the problem of simulating a two-sender classical and quantum to classical multiple-access channel (MAC). We assume that the respective encoders and decoder have access to unlimited shared randomness. This framework was first investigated by Bennett *et al.* [2] to establish a so-called 'reverse Shannon theorem' to simulate a noisy channel from a noiseless channel in the asymptotic independent and identically distributed (iid) regime. They showed that the least communication cost for this purpose is equal to the mutual information, $I(X; Y)$, between the input and output of the channel. The minimum one-shot rate for simulating a point to point classical channel was ascertained in [3]. Extensions to broadcast channels were obtained in [3], and recently extended to the quantum setting [4].

In both the point to point and broadcast channel simulation tasks, one may gain intuition on the scheme achieving the minimal communication rate as follows. Consider the case of point to point channel simulation. Since both the encoder, say Alice, and the decoder, Bob, knows the channel to be simulated, Alice can determine the channel output at her end and then compress it 'optimally' and send it to Bob using the rate limited link. Bob then just outputs the target sequence after decompressing what he received from Alice. Similar intuition also works for the broadcast channel simulation problem. However, this approach breaks down for the MAC since there are two senders involved. More specifically, although each sender knows the MAC to be simulated, they cannot "locally" simulate the channel since the input of the other sender is unknown. Hence, novel schemes are required to circumvent this technical hurdle, which we address in this work.

While there is a comprehensive literature on simulating a point to point channel (both classical and quantum, in one-shot and asymptotic iid setting) and broadcast channel ([2], [4]–[9]), only some of the corresponding results for MAC are known. In this regard, the asymptotic iid rate region for a classical MAC was previously given in [10]. Additionally, a matching converse for the more general case when shared randomness at either parties is unbounded and the decoder has access to side information correlated with the inputs such that the inputs are independent conditioned on this side information. An inner bound for MAC simulation in [10] was derived by using the so-called output-statistics of random binning (OSRB) technique of Yassaee *et al.* [8]. A matching outer bound was proven by using the continuity property of the mutual information.

In this work, we obtain the first tight one-shot rate region (tight up to the smoothing fudge terms) for simulating a classical MAC with inputs (X_1, X_2) and output Y, and the channel is represented by the conditional probability distribution $q_{Y|X_1,X_2}$. In order to simulate $q_{Y|X_1,X_2}$, encoder $\mathcal{E}_j$, for $j \in \{1, 2\}$, at sender j sends a message $M_j \in [1 : 2^{R_j}]$ to the decoder $\mathcal{D}$ over their respective noiseless links based on their individual observations and shared randomness with the decoder. We assume unlimited shared randomness $S_1(|S_1| = \infty)$ between $\mathcal{E}_1$ and $\mathcal{D}$ and $S_2(|S_2| = \infty)$ between $\mathcal{E}_2$ and $\mathcal{D}$.

Subsequent to deriving the one-shot result for purely classical MAC, we specialize our result to the asymptotic iid setting and show that it recovers [10, Theorem 1, Theorem 4] (when simplified to our model by setting the inputs X_1, X_2 to be independent, the side information Z to be trivial and both the shared randomness to infinity). We then characterize the rate region for the so-called classical scrambling quantum-inputs and classical output MAC, referred to as classical scrambling QC-MAC.

A general recipe to obtain an inner bound on the rate region is via the technique of *rejection sampling*. One of the main technical hurdles, besides unavailability of both the inputs at the encoders, preventing the import of earlier results is that this task cannot be seen as carrying out two point to point channel simulation. The core reason is due to the fact that the output must be generated correlated with both the inputs. This is resolved by defining the appropriate *auxiliary* random variables, which are quantized versions of the respective inputs such that they approximately simulate the channel.

II. Task Formulation for Fully Classical MAC

We use the standard notations of information theory from [11], [12]. We start by giving the formal definition of a 2-user MAC channel simulation code.

Definition 1 (Classical MAC simulation): An (R_1, R_2, ε) simulation code for a 2-independent user fully classical MAC $q_{Y|X_1X_2}$ with inputs $q_{X_1} \otimes q_{X_2}$ and access to unlimited shared randomness between Sender 1 $\overset{S_1}{\leftrightarrow}$ Receiver and Sender 2$\overset{S_2}{\leftrightarrow}$ Receiver, consists of:

- A pair of Encoders of form $\mathcal{E}_1 \otimes \mathcal{E}_2$, such that: $\mathcal{E}_j : \mathcal{X}_j \times \mathcal{S}_j \to [1 : 2^{R_j}]$, for $j \in \{1, 2\}$;
- Two independent noiseless rate-limited links of rate R_j, $j \in \{1, 2\}$ and;
- A decoder $\mathcal{D} : [1 : 2^{R_1}] \times \mathcal{S}_1 \times [1 : 2^{R_2}] \times \mathcal{S}_2 \to \mathcal{Y}$ s.t.;

$$\mathop{\mathbb{E}}_{q_{X_1}\otimes q_{X_2}} \left\| \mathcal{D} \circ \left(\bigotimes_{j=1}^{2} \mathcal{E}_j \right) \left\{ \bigotimes_{j=1}^{2} \left(p_{X_j} \otimes p_{S_j} \right) \right\} - q_{Y|X_1X_2} \right\|_1 \leq \varepsilon$$

III. Rejection sampling for channel simulation

In this section we give the key tool used to establish our results for one-shot fully classical and classical scrambling QC-MAC simulation. Although this tool is not new and has a quite exhaustive literature ([4], [7], [13], [14]), we state the version that is used in our proofs. We prove the achievability and the converse for our classical rejection sampling technique aka one-shot message compression or one-shot covering lemma. This can be thought of as a 'partial' one-shot analogue of the widely applicable OSRB technique developed by Yassaee *et al.* in [8] for proving alternative achievability bounds for problems in classical Shannon theory including the channel simulation task, carried out in [15]. The only known non-asymptotic one shot analogue of OSRB techniques was developed by the same authors. But the rate expressions there are in terms of empirical entropies which may be less operational than the state of art, smoothed Rényi entropies.

We first state an achievable rate for the one-shot message compression via the rejection sampling. This is nothing but an upper bound on the communication required to carry out rejection sampling. The idea is simple and useful for our purpose. The rate defining quantity is smoothed max-mutual information as defined in Definition 3 later.

Lemma 1: Let $\varepsilon > 0$ and $p_{X,Y}$ denote a bipartite probability distribution and $S \sim p_S$ be the shared randomness between sender Alice and the receiver Bob. Alice sends a message $m(x, s) \in \mathcal{M}$ to Bob, so that Bob can generate a sample $y(m, s) \overset{\varepsilon}{\sim} p_{Y|X=x}$. The minimum rate for the above task is:

$$R := \log |\mathcal{M}| \leq I^{\varepsilon}_{\max}(X; Y) + O\left(\log \log \frac{1}{\varepsilon}\right) + 1,$$

and the resulting distribution $\tilde{p}_{X,Y}$ satisfies $\|\tilde{p}_{X,Y} - p_{X,Y}\|_1 \leq 10\varepsilon$ and $\|\tilde{p}_Y - p_{Y|X=x}\|_1 \leq 10\varepsilon$.

The proof of the above lemma can be deduced from [7, Theorem 2]. We now give an 'almost' matching lower bound on the number of bits communicated in the rejection sampling algorithm for message compression. We use the word 'almost' as the lower bound that we show differs from the upper bound in the additive fudge factors and the smoothing parameter in the entropic quantity but the rate governing entropy remains the same, which is $I^{\varepsilon}_{\max}$.

Now, suppose that $p_{X,Y}$ be the given distribution, S be the shared randomness available at sender and receiver and let $\tilde{p}_{X,Y}$ be the probability distribution produced by the rejection sampling algorithm. We further have that for any given $\varepsilon > 0$, $\|p_{X,Y} - \tilde{p}_{X,Y}\|_1 \leq 10\varepsilon$. Rejection sampling algorithm produces the following Markov chain $X \to (M, S) \to Y$, where M is the classical message communicated. Lastly, we also consider $\varepsilon, \delta > 0$ as the 'smoothing' and 'continuity' parameters henceforth. One can also set $\delta = \varepsilon$. The converse for the above task (of Lemma 1) is as follows:

Lemma 2: Let $p_{X,Y}$ denote a bipartite probability distribution and $S \sim p_S$ shared randomness between the sender and the receiver. Suppose there exists a rejection sampling algorithm of Lemma 1 that outputs a resulting distribution $\tilde{p}_{X,S,M,Y}$ satisfying $\|\tilde{p}_{X,Y} - p_{X,Y}\|_1 \leq 10\varepsilon$ and $\|\tilde{p}_Y - p_{Y|X=x}\|_1 \leq 10\varepsilon$. Then for any given $\varepsilon, \delta > 0$ the rate of the rejection sampling algorithm is lower bounded by:

$$R := \log |\mathcal{M}| \geq I^{20\varepsilon+\delta}_{\max}(X; Y)_p - \log\left(\frac{1}{\varepsilon}\right) - 1$$

The proof is similar to [7, Theorem 4].

Definition 2: The max divergence between any two quantum states ρ and σ (analogously between any two probability distributions p and q on the support $\mathcal{X}$) is defined as:

$$D_{\max}(\rho||\sigma) := \log\|\sigma^{-1/2}\rho\sigma^{-1/2}\|_\infty;$$
$$D_{\max}(p||q) := \log \max_{x\in\mathcal{X}} \left[\frac{p(x)}{q(x)}\right].$$

In order to obtain the Shannon mutual information as the limit of the max mutual information in the asymptotic iid limit, we need to define their smoothed versions by replacing the distribution p (or the state ρ) by a distribution (or state) which lies in an $\varepsilon(> 0)$-ball around p (or ρ) and not perturbing q (or σ). For this smoothing purpose we define an ε (> 0)-ball as $\mathcal{B}_\varepsilon(p) := \{p' \in \mathcal{P}_\leq : \|p-p'\|_1 \leq \varepsilon\}$. For brevity, we henceforth use the notation $p \overset{\varepsilon}{\approx} q \Rightarrow \|p-q\|_1 \leq \varepsilon$ and $X \overset{\varepsilon}{\sim} p$ to denote that X has a distribution that is ε-close to p in ℓ_1 norm. Lastly, $p^{\otimes n}(x^n) := \prod_{i=1}^{n} p(x_i)$.

Definition 3: For any $\varepsilon > 0$, the ε-smoothed max-mutual information evaluated with respect to a bipartite distribution $p_{X,Y}$ is defined as:

$$I^{\varepsilon}_{\max}(X;Y)_p \quad := \inf_{p'_{X,Y} \in \mathcal{B}_\varepsilon(p_{X,Y})} D_{\max}(p'_{X,Y} \| p_X \times p_Y).$$

For $\varepsilon = 0$, we refer to the above quantity simply as $I_{\max}$.

Finally, for deriving our results for the QC-MAC in Section VI, we use the so-called coherent rejection sampling or convex split lemma (first formulated in [16]). Again, this is the core idea used to prove the one-shot measurement compression theorem in [14], which can be modified to carry out the QC-MAC simulation task, as we do here. We remark that the additive fudge term in the convex split lemma is $O(\log 1/\varepsilon)$ in contrast to $O(\log\log 1/\varepsilon)$ in the classical setting. This is because the classical rejection sampling 'fine tunes' the input to be correlated with only the accepted sample from the shared randomness whereas the convex split step correlates input with all the registers of the shared randomness.

IV. One-shot rate region (cost) of classical MAC

A. Achievability

Lemma 3: For any given $\varepsilon > 0$, there exists an $(R_1, R_2, 2\varepsilon)$ one-shot MAC simulation protocol satisfying

$$\begin{aligned} R_1 &\geq I^{\varepsilon}_{\max}(X_1;U_1)_p + O\left(\log\log\frac{1}{\varepsilon}\right) + 1; \\ R_2 &\geq I^{\varepsilon}_{\max}(X_2;U_2)_p + O\left(\log\log\frac{1}{\varepsilon}\right) + 1, \end{aligned}$$

such that there exists auxiliary random variables U_1, U_2 and a distribution $p_{X_1,X_2,U_1,U_2,Y} := q_{X_1} p_{U_1|X_1} q_{X_2} p_{U_2|X_2} p_{Y|U_1,U_2}$ satisfying

$$\sum_{u_1,u_2} p_{X_1,X_2,U_1,U_2,Y}(x_1,x_2,u_1,u_2,y) = q_{X_1,X_2,Y}(x_1,x_2,y),$$

The above region is evaluated with this p and the cardinalities of U_1, U_2 are finite.

Proof: Let $q_{X_1} \otimes q_{X_2}$ be a given input distribution and fix an $\varepsilon > 0$. We will use the rejection sampling algorithm (with access to p_{X_1,X_2,U_1,U_2}) of Lemma 1 twice to come up with two encoder-decoder pairs to simulate $q_{Y|X_1,X_2}$ via $\{U_j\}_{j=1}^2$. The key idea is to use the least amount of communication to facilitate the decoder to sample $\{U_j\}_{j=1}^2 \overset{\varepsilon}{\sim} p_{U_j|X_j=x_j}$. Towards this end, let the encoder-decoder pair for sender-1 is denoted by $(\mathcal{E}_1, \mathcal{D}_1)$ and that for sender-2 by $(\mathcal{E}_2, \mathcal{D}_2)$. These encoder-decoder pairs produces a joint distribution on X_1, X_2, U_1, U_2, Y that is close to the distribution p mentioned in the theorem statement.

- **Sender-**j for $j \in \{1,2\}$: The shared randomness between the pair $(\mathcal{E}_j, \mathcal{D}_j)$ is distributed according to the marginal $p_{U_j}(u_j) := \Sigma_{x_j} q_{X_j}(x_j) p_{U_j|X_j}(u_j|x_j)$. Using this, $(\mathcal{E}_j, \mathcal{D}_j)$ perform rejection sampling (Lemma 1) so that the output distribution of U_j from $\mathcal{D}_j$ is denoted by $p^{algo}_{U_j} \overset{\varepsilon}{\approx} p_{U_j|X_j=x_j}$.
- **Decoding:** After having decoded U_1, U_2 from $\mathcal{D}_1, \mathcal{D}_2$ the receiver finally generates $Y \sim p_{Y|U_1,U_2}$. Thus, the overall decoder $\mathcal{D} = p_{Y|U_1,U_2} \circ (\mathcal{D}_1 \otimes \mathcal{D}_2)$.

Thus, our algorithm results in the overall distribution

$$p^{algo}_{X_1,X_2,U_1,U_2,Y} = q_{X_1} q_{X_2} p^{algo}_{U_1} p^{algo}_{U_2} p_{Y|U_1,U_2}.$$

Note that $p^{algo}_{U_j}(u_j) \approx p_{U_j|X_j}(u_j|x_j)$ for $j \in \{1,2\}$ and $X_j = x_j$ with probability $q_{X_j}(x_j)$. Finally, we need to show that $\mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p^{algo}_{Y|X_1,X_2} - q_{Y|X_1,X_2}\|_1 \leq 2\varepsilon$. This follows by the following chain of inequalities:

$$\begin{aligned}
&\mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p^{\text{algo}}_{Y|X_1,X_2} - q_{Y|X_1,X_2}\|_1 \\
&\leq \mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p^{\text{algo}}_{Y|X_1,X_2} - p_{Y|X_1,X_2}\|_1 \\
&\quad + \mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p_{Y|X_1,X_2} - q_{Y|X_1,X_2}\|_1 \\
&\leq \mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p^{algo}_{Y|X_1,X_2} - p_{Y|X_1,X_2}\|_1 + \|p_{X_1,X_2,Y} - q_{X_1,X_2,Y}\|_1 \\
&= \mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p^{algo}_{Y|X_1,X_2} - p_{Y|X_1,X_2}\|_1 \\
&\leq \mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \Big\| \sum_{u_1,u_2} \Big[p^{\text{algo}}_{U_1}(u_1) p^{\text{algo}}_{U_2}(u_2) P_{Y|U_1=u_1,U_2=u_2} \\
&\quad - p_{U_1|X_1}(u_1|x_1) p_{U_2|X_2}(u_2|x_2) P_{Y|U_1=u_1,U_2=u_2} \Big] \Big\|_1 \\
&\leq \mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p^{\text{algo}}_{U_1} p^{algo}_{U_2} - p_{U_1|X_1} p_{U_2|X_2}\|_1 \\
&\leq \mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p^{\text{algo}}_{U_1} p^{algo}_{U_2} - p^{algo}_{U_1} p_{U_2|X_2}\|_1 \\
&\quad + \|p^{\text{algo}}_{U_1} p_{U_2|X_2} - p_{U_1|X_1} p_{U_2|X_2}\|_1 \\
&\leq \mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p^{\text{algo}}_{U_1}\|_1 \|p^{algo}_{U_2} - p_{U_2|X_2}\|_1 \\
&\quad + \mathop{\mathbb{E}}_{q_{X_1} \otimes q_{X_2}} \|p_{U_2|X_2}\|_1 \|p^{\text{algo}}_{U_1} - p_{U_1|X_1}\|_1 \leq 2\varepsilon.
\end{aligned}$$

We have thus shown that there exists an $(R_1, R_2, 2\varepsilon)$ code for simulating $q_{Y|X_1,X_2}$. We can now optimize over all U_1, U_2 satisfying $(X_1, X_2) \to (U_1, U_2) \to Y$ (for the fixed $q_{X_1} \times q_{X_2}$) and we take the union over all the distribution $p_{X_1,X_2,U_1,U_2,Y}$ satisfying the conditions of Lemma 3, calling the resultant region as the optimal achievable region. Thus, for a fixed input the above mentioned region is an inner bound for simulating a classical MAC. ■

B. Converse

Lemma 4: Let $\varepsilon \in (0, 1/2)$. Then, a (R_1, R_2, ε) MAC simulation algorithm satisfies

$$R_j \geq I^{4\varepsilon}_{\max}(X_j;U_j)_p - \log\left(\frac{1}{\varepsilon}\right), j \in \{1,2\},$$

for some distribution $p_{X_1,X_2,U_1,U_2,Y} = q_{X_1} q_{X_2} p_{U_1|X_1} \times p_{U_2|X_2} p_{Y|U_1,U_2}$ such that

$$\|p_{X_1,X_2,Y} - q_{X_1,X_2,Y}\|_1 \leq 5\varepsilon. \quad (1)$$

We defer the proof in the full version [17]. Although we assume that the amount of shared randomness is unlimited, we only use as much shared randomness as required for optimal message compression. In this sense, our simulation protocol is nearly optimal. The assumption that shared randomness is unlimited ensures that the sum rate constraints are trivial and the rate region is as given by Lemma 3 and 4. Furthermore, if one of the senders is trivial (does not send anything, that is, considering the task of a point to point channel simulation) the optimal $U = Y$. We henceforth recover the asymptotically optimal point-point channel simulation rate of $I(X;Y)_{q_{XY}}$ as shown in [3]. Moreover, for the point to point case our technique can be straight away extended to obtain the universal channel simulation by identifying optimal $U = Y$. However, the universality is not straightforward for MAC as after maximizing over the inputs (the worst case scenario) the optimal U_1, U_2 for R_1 need not be optimal for R_2 and vice-versa. Alternatively, the optimal (U_1, U_2) pair from the worst-case (R_1, R_2) pair need not satisfy $(X_1, X_2) \to (U_1, U_2) \to Y$. It is this subtlety that finally Y should be generated jointly from U_1, U_2, which prevents the analysis of MAC simulation as simulation of two independent point-point channels.

V. Asymptotic IID rate region

The inner bound can be straight away extended to obtain the asymptotic iid rate region, since the n-fold extension of the input and the auxiliary random variables will be iid. The asymptotic equipartition property (AEP [18]) for $I^{\varepsilon}_{\max}$ gives:

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} I^{\varepsilon}_{\max}(X^n_j, U^n_j)_{p^n} = I(X_j; U_j)_p$$

and leads to the asymptotically optimal inner bound $R_j \geq I(X_j; U_j)_p$. Note that obtaining the asymptotically optimal outer bound is not so straight forward as the n-fold extension of the random variable U need not be iid. So, we prove a *weak converse*.

Proof: Consider the n-fold extension of the simulation code and the code induced distribution $p^n_{X^n_1, U^n_1, X^n_2, U^n_2, Y^n} := q^{\otimes n}_{X_1} q^{\otimes n}_{X_2} p_{U^n_1|X^n_1} p_{U^n_2|X^n_2} p_{Y^n|U^n_1, U^n_2}$ in Lemma 4. Consider the following inequalities on the rate of the protocol:

$$\begin{aligned}
nR_j &\geq I^{\varepsilon}_{\max}(X^n_j; U^n_j)_{p^n} - \log \frac{1}{\varepsilon} \\
&\overset{(a)}{=} I_{\max}(X^n_j; U^n_j)_{p'^n} - \log \frac{1}{\varepsilon} \\
&\overset{(b)}{\geq} I(X^n_j; U^n_j)_{p'^n} - \log \frac{1}{\varepsilon} \\
&\overset{(c)}{\geq} I(X^n_j; U^n_j)_{p^n} - 6\varepsilon \log |\mathcal{X}_j|^n - 8h_2(\varepsilon) - \log \frac{1}{\varepsilon} \\
&\overset{(d)}{\geq} nI(X_j; U_j)_p - 6\varepsilon \log |\mathcal{X}_j|^n - 8h_2(\varepsilon) - \log \frac{1}{\varepsilon} \\
\Rightarrow R_j &\geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left[\frac{nI(X_j; U_j)_p - 6\varepsilon \log |\mathcal{X}_j|^n - g(\varepsilon)}{n} \right] \\
\Rightarrow R_j &\geq I(X_j; U_j)_p,
\end{aligned}$$

where (a) holds as $p' \in \mathcal{B}^{\varepsilon}(p)$ be the optimizer for $I^{\varepsilon}_{\max}$; (b) holds by the fact the $I^{\varepsilon}_{\max}(X;Y)_p \geq I(X;Y)_p - O\left(\log \frac{1}{\varepsilon}\right)$ for any joint distribution p_{XY} [19]; (c) follows from continuity of mutual information [20, Lemma 4] with $g(\varepsilon) = 8h_2(\varepsilon) + \log \frac{1}{\varepsilon}$; (d) follows by $I(X^n_j; U^n_j) = \sum_{i=1}^n \left[H(X_{j,i}) - H(X_{j,i}|U^n_j)\right] \geq \sum_{i=1}^n \left[H(X_{j,i}) - H(X_{j,i}|Uj, i)\right] \geq \sum_{i=1}^n I(X_{j,i}; U_{j,i})$, using the facts that the input is a product distribution and conditioning reduces the entropy and lastly, following the single letterization steps of time sharing from the iid converse in [10, Theorem 3] (where we abuse the notation slightly to denote the time shared U_j also as U_j). We thus recover the asymptotically optical region of [10, Theorem 1, Theorem 3]. ■

VI. Rate region (cost) of quantum-classical MAC

In this section, we consider a MAC with the two input quantum states in tensor product (independent) and the output as a classical state (random variable or equivalently a probability distribution). Generally, a device which takes input as a quantum state and outputs a probability vector is thought of as a measurement device. Hence we can think of our 2-quantum input and 1-classical output QC MAC as a measurement channel. There can be several configurations of this models. We consider the following model:

Classical scrambling QC MAC (CS-QC MAC): Channel first does a 'separable' measurement on two inputs and then scrambles them according to $q_{Y|X_1,X_2}$. We refer to such channels as "*classical scrambling*" channels, denoted as

$$\mathcal{N}^{AB\to Y}_{CS} := \sum_{x_1, x_2, y} q_{Y|X_1,X_2}(y|x_1,x_2) \, |y\rangle\langle y|^Y \otimes \left(\Lambda^A_{x_1} \otimes \Gamma^B_{x_2}\right) \tag{2}$$

where $\{\Lambda_{x_1}\}_{x_1}$ and $\{\Gamma_{x_2}\}_{x_2}$ are complete POVMs acting on Hilbert spaces $\mathcal{H}^{A_1}$ and $\mathcal{H}^{A_2}$ respectively. The same model of a QC-channel, termed as a POVM having a separable decomposition with stochastic integration was proposed in [21].

Definition 4 (CS-QC MAC with feedback simulation): An (R_1, R_2, ε) simulation code for a 2-independent user CS-QC MAC given in (2) with inputs as pure states obtained by purification of $\Lambda(\rho_A) \otimes \Gamma(\sigma_B)$ and access to unlimited shared randomness between Sender 1 $\overset{S_1}{\leftrightarrow}$ Receiver and Sender 2 $\overset{S_2}{\leftrightarrow}$ Receiver, consists of:

- A pair of Encoders of form $\mathcal{E}_1 \otimes \mathcal{E}_2$, such that: $\mathcal{E}_j : \mathcal{X}_j \otimes \mathcal{S}_j \to [1 : 2^{R_j}]$, for $j \in \{1, 2\}$;
- Two separate noiseless rate-limited classical links of rate R_j, $j \in \{1, 2\}$ and;
- A decoder $\mathcal{D} : [1 : 2^{R_1}] \times \mathcal{S}_1 \times [1 : 2^{R_2}] \times \mathcal{S}_2 \to \mathcal{Y}$ such that

$$\left\| \mathcal{D} \circ \left(\bigotimes_{j=1}^{2} \mathcal{E}_j \right) \left(\Lambda(\rho_A) \otimes \Gamma(\sigma_B) \otimes \bigotimes_{j=1}^{2} p_{S_j} \right) - q_{Y|X_1X_2} \right\|_1 \leq \varepsilon.$$

A. Achievability

Definition 5: The quantum smoothed max-mutual information is defined as

$$I^{\varepsilon}_{\max}(A;B)_{\rho} := \inf_{\rho'^{AB} \in \mathcal{B}_{\varepsilon}(\rho^{AB})} D_{\max}(\rho'^{AB} || \rho^A \otimes \rho^B), \tag{3}$$

where $D_{\max}$ and $\mathcal{B}_{\varepsilon}$ are defined in Definition 2.

Theorem 6.1: For any given $\varepsilon > 0$, there exists an (R_1, R_2, ε) one-shot CS-QC MAC simulation protocol satisfying

$$R_1 \geq I_{\max}^{\frac{\varepsilon}{2}}(U_1; R_A)_\eta + 7\log\left(\frac{2}{\varepsilon}\right) + 1; \quad (4)$$

$$R_2 \geq I_{\max}^{\frac{\varepsilon}{2}}(U_2; R_B)_\eta + 7\log\left(\frac{2}{\varepsilon}\right) + 1, \quad (5)$$

where $\eta^{YR_AR_BU_1U_2X_1X_2}$ is classical quantum state defined as

$$\eta := \sum_{u_1,u_2} p_{Y|U_1U_2}(y|u_1,u_2)\,|y\rangle\langle y|^Y \otimes |x_1\rangle\langle x_1|^{X_1} \otimes |x_2\rangle\langle x_2|^{X_2} \otimes p_{U_1|X_1}(u_1|x_1)p_{X_1}(x_1)\,|u_1\rangle\langle u_1|^{U_1} \otimes p_{U_2|X_2}(u_2|x_2)p_{X_2}(x_2)\,|u_2\rangle\langle u_2|^{U_2} \otimes \left\{\mathcal{I}^{R_A} \otimes \Lambda_{x_1}^{A}\right\}(\phi_\rho^{R_AA}) \otimes \left\{\mathcal{I}^{R_B} \otimes \Gamma_{x_2}^{B}\right\}(\psi_\sigma^{R_BB}), \quad (6)$$

s.t. $I(R_A, R_B; Y|U_1, U_2)_\eta = 0$ and $\mathcal{N}(\phi_\rho \otimes \psi_\sigma) = \eta^{R_AR_BY}$.

Proof: The proof for the inner bound follows by applying the so-called one-shot measurement compression theorem with feedback [14, Theorem 1] twice, for each sender. The achievability protocol is constructed using:

- **Encoding:** Both the senders locally obtain the measurement outcome X_j, and then generate the classical auxiliary random variables U_j by post processing X_j with the dephasing map $\mathcal{I} \otimes \mathcal{C}_j^{X_j} \to U_j$. The map $\mathcal{C}_j^{X_j \to U_j}\left(\sum_x p_{X_j}(x)\,|x\rangle\langle x|^{X_j}\right) := \sum_{u,x} p_{X_j}(x_j)p_{U_j|X_j}(u|x)\,|u\rangle\langle u|^{U_j}$, is a measurement channel that measures the state on X_j in an o.n.b. $|u\rangle^{U_j}$ and outputs the conditional distribution $p_{U_j|X_j}$. Thus, we can the extended measurement operators and an extension η of the channel output, respectively, as:
$$\tilde{\Lambda}^{A\to U_1} \otimes \tilde{\Gamma}^{B\to U_2} := (\mathcal{C}_1 \circ \Lambda)^{A\to U_1} \otimes (\mathcal{C}_2 \circ \Gamma)^{B \to U_2}$$
$$\Rightarrow \eta = \sum_{y,u_1,u_2} p_{Y|U_1,U_2}(u_1,u_2)\,|y\rangle\langle y|^Y \otimes (\tilde{\Lambda}_{u_1} \otimes \tilde{\Gamma}_{u_2})(\phi_\rho \otimes \psi_\sigma). \quad (7)$$
Hence, the *Encoders* $\mathcal{E}_j : A_jS_jU_jU_j \to [1 : 2^{R_j}]$ (with the inputs $A_1 := A$ and $A_2 := B$) are the encoders used in the measurement compression protocol with feedback [14, Theorem 1].
- **Decoders:** $\mathcal{D}_j : [1 : 2^{R_j}] \otimes S_j \to \mathcal{U}_j$ are the decoders used in measurement compression [14, Theorem 1] to recover U_1, U_2 by "preserving" correlations with R_A, R_B. Then final decoder output is $Y \sim q_{Y|U_1,U_2}$ and thus $\mathcal{D} := p_{Y|U_1,U_2} \circ (\mathcal{D}_1 \otimes \mathcal{D}_2)$.

Analysis technique: The one-shot measurement compression theorem with feedback [14, Theorem 1] approximately simulates the measurement outcomes of $\{\tilde{\Lambda} \otimes \tilde{\Gamma}\}$ at the receiver, who then scrambles these simulated outcomes (U_1, U_2) according to $p_{Y|U_1,U_2}$ (or stochastically generates Y) to approximately simulate the QC-MAC given by equation 2. The communication cost is given by equation 4, incurred from measurement compression. Alternately, the above rates can also be derived using the one-shot state splitting technique illustrated in [6, Theorem 10]. ■

B. Converse

Theorem 6.2: For a given $\varepsilon > 0$ and a classical scrambling QC-MAC given by Equation 2 with inputs $\Lambda(\rho^A), \Gamma(\sigma^B)$ and their respective purifications $|\phi_\rho\rangle^{R_AAX_1}, |\psi_\sigma\rangle^{R_BBX_2}$, any simulation code satisfies the following outer bound

$$R_1 \geq I_{\max}^{\varepsilon}(R_A; U_1)_\tau - O(\log \tfrac{1}{\varepsilon}), \quad (8)$$
$$R_2 \geq I_{\max}^{\varepsilon}(R_B; U_2)_\tau - O(\log \tfrac{1}{\varepsilon}), \quad (9)$$

evaluated with respect to the state $\tau^{R_AR_BU_1U_2YX_1X_2}$:

$$\tau := \sum_{y,u_1,u_2,x_1,x_2} p_{Y|U_1,U_2}(y|u_1,u_2) \prod_{j=1}^{2}\left\{p_{X_j}(x_j)p_{U_1|X_1}(u_j|x_j)\right\}|y\rangle\langle y| \otimes |x_1x_2\rangle\langle x_1x_2| \otimes |u_1\rangle\langle u_1| \otimes p_{U_2|X_2}(u_2|x_2)\,|u_2\rangle\langle u_2| \otimes \tau_{u_1,u_2}^{R_AR_BY}$$

such that

$$\|\tau^{R_AR_BY} - \left[\mathcal{I}^{R_AR_B} \otimes \mathcal{N}^{AB\to Y}\right](\phi_\rho^{R_AA} \otimes \psi_\sigma^{R_BB})\|_1 \leq \varepsilon\,.$$

The proof of this theorem is deferred in the full version [17].

C. Asymptotic iid rate region

Corollary 1: Consider the classical scrambling QC-MAC $\mathcal{N}^{AB\to Y}$, given by Equation 2 and inputs ρ^A, σ^B with their respective purifications $|\phi_\rho\rangle^{R_AA}, |\psi_\sigma\rangle^{R_BB}$. The rate region for simulating $\mathcal{N}^{AB\to Y}$ using infinite shared randomness between each sender-receiver pair and classical communication over links of (R_1, R_2) is closure of the regions described by:

$$R_1 \geq I(U_1; R_A), \quad R_2 \geq I(U_2; R_B) \quad (10)$$

where the above mutual information are calculated with respect to the state $\tau^{R_AR_BU_1U_2YX_1X_2}$:

$$\tau := \sum_{u_1,u_2x_1,x_2,y} p_{Y|U_1,Y_2}p_{X_1}(x_1)p_{X_2}(x_2)\,|y\rangle\langle y| \otimes |x_1x_2\rangle\langle x_1x_2| \otimes p_{U_1|X_1}(u_1|x_1)p_{U_2|X_2}(u_2|x_2)\,|u_1u_2\rangle\langle u_1u_2| \otimes \tau_{u_1,u_2}^{R_AR_BY}$$

such that $\tau^{R_AR_BY} = [\mathcal{I}^{R_AR_B} \otimes \mathcal{N}^{AB\to Y}](\phi_\rho^{R_AA} \otimes \psi_\sigma^{R_BB})$.

The proof of this lemma can be found in the full version of this work [17].

VII. Conclusion and Open Problems

In this work, we have provided a first tight characterization (via almost matching inner and outer bounds) for simulating a 2-independent user classical MAC, with unlimited shared randomness. Further, we have derived an analogous extension to the classical scrambling QC-MAC and also provided its asymptotic iid characterization. The quest to make our protocols independent of the inputs, also known as the universal channel simulation and bound the cardinality of auixiliaries, are a part of ongoing research. Working with classical-to-quantum and fully quantum MAC with entanglement assistance is a further interesting open problem.

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