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Algorithms & Complexity Theory Seminar Series
Oxford, 16 May 2019

Semidefinite programming hierarchies for quantum information

Mario Berta

[SIAM Journal on Optimization '16] with Fawzi and Scholz
[arXiv:1810.12197] with Borderi, Fawzi, and Scholz

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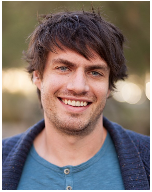
Theory of Quantum Information at Imperial



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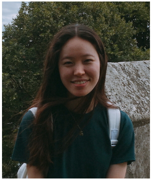


Hyejung Jee
PhD Controlled Quantum Dynamics

Navneeth Ramakrishnan
Imperial President's PhD

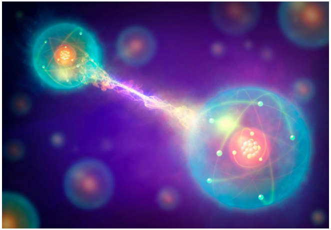
Samson Wang
PhD Quantum Systems Engineering

Quantum Information Processing

- The theory of information processing depends on underlying physical laws.
- Quantum information theory is based on (non-relativistic) quantum mechanics.

Research over the past two decades has shown:

Quantum information is in general fundamentally different from classical information.



Entangled quantum particles

- Quantum information is largely rooted in physics and thus often uses techniques motivated from physics.

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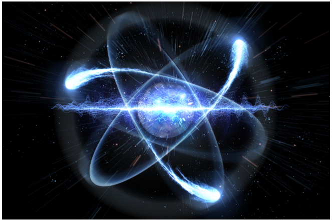
Theoretical Computer Science for Quantum Information

Basic idea:

Many quantities of interest in information processing can be phrased as bilinear maximization programs $\Rightarrow$ make use of this structure.

- Winning probability of two-player games (CS) — violation of Bell inequalities (Physics)
- Information theory, e.g., success probability of error correcting codes
- Cryptography, e.g., cheating probability of adversaries
- Communication complexity
- Completely positive semidefinite cone, etc.



Quantum technologies @Imperial

Optimization methods for quantum information processing:

Approximate bilinear maximization programs via semidefinite programming hierarchies!

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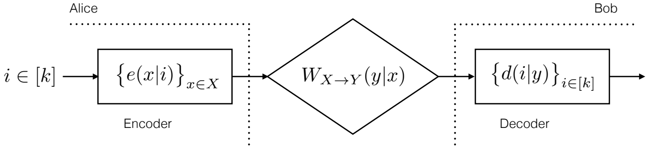
This Talk

General approach:

Goal is to quantify the difference between classical and quantum information for operational problems — via semidefinite programming hierarchies.

- Develop by means of concrete example — noisy channel coding:



Noisy channel $W_{X \to Y}$ mapping X to Y with transition probability $W_{X \to Y}(y|x)$.

- Outline:
  0. Classical channel coding
  1. Quantum-assisted channel coding
  2. Quantum channel coding

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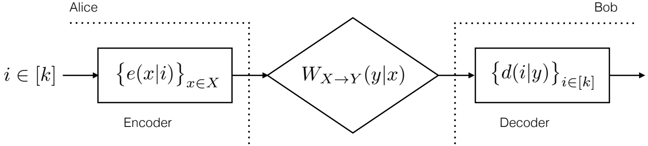
Part 0: Noisy Channel Coding



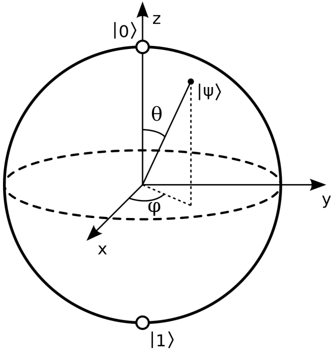
- The goal is to send k bits using $W_{X\to Y}$ while maximizing the success probability for decoding:

$$
\begin{aligned}
p(W,k) = \underset{e,d}{\text{maximize}} \quad & \frac{1}{2^k}\sum_{x,y,i} W_{X\to Y}(y|x)\,d(i|y)\,e(x|i) \\
\text{subject to} \quad & \sum_x e(x|i) = 1\ \forall i \in [2^k], \quad \sum_i d(i|y) = 1\ \forall y \in Y \\
& 0 \le e(x|i) \le 1, \quad 0 \le d(i|y) \le 1
\end{aligned}
$$

- Information-theoretic approach to error correction: Bilinear optimisation with linear constraints.

Quantum Assistance: Bits versus Qubits

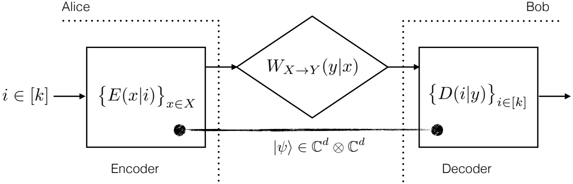
- Classical information unit:
  Bits take values 0 or 1 with certain probabilities
- Quantum information unit:
  Qubits take values $|\psi\rangle$ on the Bloch sphere $S^2 \subset \mathbb{R}^3$
- Quantum effects such as uncertainty principle for measurements or entanglement from quantum correlations



Quantum state:

For d degrees of freedom general quantum state given by unit vector $|\psi\rangle \in \mathbb{C}^d$.

Part 1: Quantum-Assisted Channel Coding



- Use $W_{X\to Y}$ and quantum assistance — that is to be optimized over:

$$p^*(W,k) = \underset{|\psi\rangle\in\mathbb{C}^d, E, D}{\text{maximize}} \quad \frac{1}{2^k}\sum_{x,y,i} W_{X\to Y}(y|x)\langle\psi|E(x|i)\otimes D(i|y)|\psi\rangle$$

$$\text{subject to} \quad \sum_x E(x|i) = \mathrm{id}_{d\times d}\ \forall i\in[2^k], \quad \sum_i D(i|y) = \mathrm{id}_{d\times d}\ \forall y \in Y$$

$$0 \preceq E(x|i) \preceq \mathrm{id}_{d\times d}, \quad 0 \preceq D(i|y) \preceq \mathrm{id}_{d\times d}, \quad \||\psi\rangle\| = 1$$

- Versus classical value: $p(W,k) = \max_{e,d} \frac{1}{2^k}\sum_{x,y,i} W_{X\to Y}(y|x)d(i|y)e(x|i)$.

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Quantum-Assisted Channel Coding (continued)

Goal is to understand the possible separation:

$p(W,k) \leq p^*(W,k) \leq ?$ Is this even computable?

- Example separation for $d = 2$ quantum assistance [Prevedel *et al.* '11]:

$$W = \begin{pmatrix} 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 0 \end{pmatrix} \text{ has } p(W,2) = \frac{5}{6} \approx 0.833 < 0.902 \approx \frac{2+2^{-1/2}}{3} \leq p^*(W,2).$$

- Also optimal for $d = 2$ quantum assistance [Hemenway *et al.* '13] but remains unclear what $p^*(W,2) = ?$
- Lower bounds on $p^*(W,k)$ via feasible points and, e.g., see-saw optimisation.

Quantum-Assisted Channel Coding (continued)

$$
\begin{aligned}
p^*(W,k) = \underset{|\psi\rangle\in\mathbb{C}^d, E, D}{\text{maximize}} \quad & \frac{1}{2^k}\sum_{x,y,i} W_{X\to Y}(y|x)\langle\psi|E(x|i)\otimes D(i|y)|\psi\rangle \\
\text{subject to} \quad & \sum_x E(x|i) = \mathrm{id}_{d\times d}\ \forall i\in[2^k], \quad \sum_i D(i|y) = \mathrm{id}_{d\times d}\ \forall y\in Y \\
& 0 \preceq E(x|i) \preceq \mathrm{id}_{d\times d}, \quad 0 \preceq D(i|y) \preceq \mathrm{id}_{d\times d}, \quad \||\psi\rangle\| = 1
\end{aligned}
$$

- See-saw based lower bounds on $p^*(W,k)$ lead to semidefinite programs.

Semidefinite programs (SDP):

Optimization of a linear objective function over the intersection of the cone of positive semidefinite matrices with an affine space.
$\Rightarrow$ solved efficiently in terms of the size of the matrices and the approximation error

- Question: How to generate upper bounds on $p^*(W,k)$?
- Answer: Non-commutative version of sum-of-squares semidefinite programming hierarchies of [Lasserre '01] and [Parrilo '03]!

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SDPs for Quantum-Assistance

Asymptotically converging hierarchy of SDP relaxations:

Motivated by [Navascues *et al.* '07 '08] [Doherty *et al.* '08] [Pironio *et al.* '10] we give

$$p^*(W,k) \le \mathrm{sdp}_n(W,k) \le \cdots \le \mathrm{sdp}_1(W,k) \text{ with } p^*(W,k) = \lim_{n\to\infty} \mathrm{sdp}_n(W,k).$$

- For our example channel W the first level gives

$$p^*(W,2) \le \mathrm{sdp}_1(W,2) \approx 0.908 = \frac{1}{2} + \frac{1}{\sqrt{6}}$$

  versus $p(W,2) = \frac{5}{6} \approx 0.833$ and $d = 2$ lower bound $p^*(W,2) \ge 0.902$.
  Then found $d = 4$ lower bound $p^*(W,2) \ge \frac{1}{2} + \frac{1}{\sqrt{6}}$!

- Follow-up [Barnam and Fawzi '18] give rounding for linear program $\mathrm{lp}(W,k)$ as

$$p(W,k) \le p^*(W,k) = \mathrm{sdp}_\infty(W,k) \le \mathrm{sdp}_1(W,k) \le \mathrm{lp}(W,k) \le \frac{1}{1-e^{-1}} \cdot p(W,k)$$

- Better than $(1-e^{-1})$-approximation NP-hard and simple polynomial-time algorithms for classical error correction codes that achieve this approximation.

SDPs for Quantum-Assistance (continued)

- General form quantum information [B. *et al.*, SIAM Journal on Optimization '16]:

$$
\begin{aligned}
p^*[A,\mathcal{G},\mathcal{K}] = \underset{|\psi\rangle\in\mathcal{H},E_\alpha,D_\beta}{\text{maximize}} \quad & \sum_{\alpha,\beta} A_{\alpha,\beta}\langle\psi|E_\alpha D_\beta|\psi\rangle \\
\text{subject to} \quad & E_\alpha D_\beta = D_\beta E_\alpha \quad \forall(\alpha,\beta)\in[N]\times[M] \\
& g(E_1,\dots,E_N)\succeq 0\ \forall g\in\mathcal{G} \\
& k(D_1,\dots,D_M)\succeq 0\ \forall k\in\mathcal{K}
\end{aligned}
$$

  with normalized $|\psi\rangle\in\mathcal{H}$ (Hilbert space), Hermitian bounded operators E_α, D_β on $\mathcal{H}$, and sets of affine constraints $\mathcal{G},\mathcal{K}$.

- Proof of asymptotic convergence

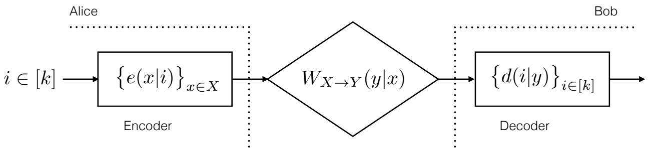
$$
p^*(A,\mathcal{G},\mathcal{K}) = \lim_{n\to\infty} \mathrm{sdp}_n(A,\mathcal{G},\mathcal{K})
$$

  via non-commutative Positivstellensatz [Helton and McCullough '04] and purifications in C^*-algebras [Woronowicz '73].

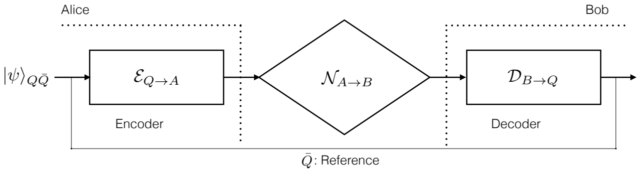
- Finite convergence unclear, related to deep problems in operator algebra theory: Connes' embedding conjecture.

Part 2: Quantum Channel Coding

- Classical channel coding (with or without quantum assistance):



- Make message and channel itself quantum — quantum channel coding:



Quantum Channel Coding (continued)

- The goal is to send k qubits using quantum channel $\mathcal{N}_{A\to B}$ while maximizing the quantum channel fidelity (uniform message quantum success probability)

$$F(\mathcal{N},k) = d_A d_B \cdot \underset{E,D}{\text{maximize}} \quad \mathrm{Tr}\left[\left(J^{\mathcal{N}}_{AB} \otimes \Phi_{Q\bar{Q}}\right)\left(E_{AQ} \otimes D_{B\bar{Q}}\right)\right]$$
$$\text{subject to} \quad E_{AQ} \succeq 0, \quad E_A = \frac{1_{d_A\times d_A}}{d_A}$$
$$D_{B\bar{Q}} \succeq 0, \quad D_B = \frac{1_{d_B\times d_B}}{d_B}$$

  with $\Phi_{Q\bar{Q}}$ the 2^k dimensional maximally entangled state and $J^{\mathcal{N}}_{\bar{A}B}$ the Choi state of $\mathcal{N}_{A\to B}$.

- Information-theoretic approach to quantum error correction:
  Bilinear optimization with matrix valued variables and linear constraints.

Noisy Intermediate-Scale Quantum (NISQ) technology:

Near-term quantum hardware is only a few qubits in size and inherently noisy, in particular quantum memory (in stark contrast to classical memory!).

SDPs for Quantum Optimization

$$F(\mathcal{N},k) = d_A d_B \cdot \underset{E,D}{\text{maximize}} \quad \mathrm{Tr}\left[\left(J^{\mathcal{N}}_{AB} \otimes \Phi_{Q\bar{Q}}\right)\left(E_{AQ} \otimes D_{B\bar{Q}}\right)\right]$$

$$\text{subject to} \quad E_{AQ} \succeq 0, \quad E_A = \frac{1_{d_A \times d_A}}{d_A}$$

$$D_{B\bar{Q}} \succeq 0, \quad D_B = \frac{1_{d_B \times d_B}}{d_B}$$

- See-saw based lower bounds on $F(\mathcal{N},k)$ lead to semidefinite programs, e.g., explored in [Reimpell & Werner '05]. Upper bounds on $F(\mathcal{N},k)$?

Non-commutative sum-of-squares hierarchy of SDP relaxations:

Motivated by [Doherty *et al.* '02] we give

$$F(\mathcal{N},k) \leq \mathrm{sdp}_n(\mathcal{N},k) \leq \cdots \leq \mathrm{sdp}_1(\mathcal{N},k) \text{ with (slow) finite convergence}$$

$$\mathrm{sdp}_n(\mathcal{N},k) - F(\mathcal{N},k) \leq \frac{\mathrm{poly}\left(d_A, d_B, 2^k\right)}{\sqrt{n}}$$

- Convergence proof information-theoretic based on quantum de Finetti theorems with linear constraints.

SDPs for Quantum Optimization (continued)

- General form quantum information [B. *et al.* arXiv:1810.12197]:

$$
\begin{aligned}
F(G,\Lambda,\Gamma,X,Y) = \underset{W}{\text{maximize}} \quad & \mathrm{Tr}\big[G_{AB}(W_A \otimes W_B)\big] \\
\text{subject to} \quad & W_A \succeq 0, \quad W_B \succeq 0, \quad \mathrm{Tr}(W_A) = \mathrm{Tr}(W_B) = 1 \\
& \Lambda_{A\to C_A}(W_A) = X_{C_A}, \quad \Gamma_{B\to C_B}(W_B) = Y_{C_B}
\end{aligned}
$$

  where G_{AB} is a matrix, $\Lambda_{A\to C_A}$ and $\Gamma_{B\to C_B}$ are linear maps and X_{C_A} and Y_{C_B} are fixed matrices defining affine subspaces.

- Approximating $F(G,\Lambda,\Gamma,X,Y)$ or already $F(\mathcal{N},k)$ encodes so-called quantum separability problem $\Rightarrow$ strong hardness results [Gharibian '10] [Harrow *et al. '19*] and slow convergence is expected (versus classical setting!).

Ongoing work:

Identify settings in quantum information that allow for faster analytical convergence, e.g., classical-quantum channels — other methods via ε-nets / polynomials needed.

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Certifying Optimality

Rank loop conditions:

We give sufficient condition for exact convergence $F(\mathcal{N}, k) = \mathrm{sdp}_n(\mathcal{N}, k)$ but this requires finding low-rank SDP solutions via rank minimization heuristics [Fazel '02].

- Example numerics: Uniform noise modelled by qubit depolarizing channel

$$\mathcal{N}_{\mathrm{Dep}(p)} : |\psi\rangle\langle\psi| \mapsto p \cdot \mathrm{Uniform}_2 + (1-p) \cdot |\psi\rangle\langle\psi| \quad \text{with } p \in [0, 4/3].$$

$\Rightarrow$ What is the optimal quantum code for storing $k = 1$ qubit in $N = 5$ noisy qubits:

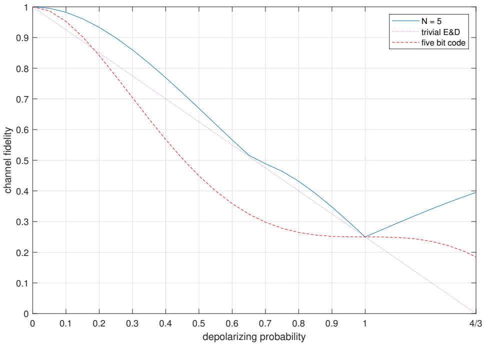
$$p\left(\mathcal{N}_{\mathrm{Dep}(p)}^{\otimes 5}, 2\right) = ?$$

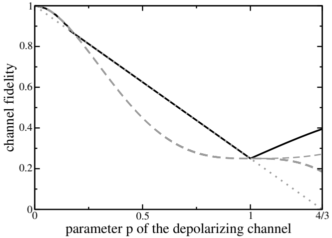
- Analytical [Bennett *et al.* '96] as well as numerical see-saw type [Reimpell & Werner '05] lower bounds available, what about upper bounds? Our work gives

$$\mathrm{sdp}_n\left(\mathcal{N}_{\mathrm{Dep}(p)}^{\otimes 5}, 2\right) \geq p\left(\mathcal{N}_{\mathrm{Dep}(p)}^{\otimes 5}, 2\right).$$

Preliminary Numerics

- Exploiting symmetries for analytical dimension reduction, we calculated the first level $\mathrm{sdp}_n\left(\mathcal{N}_{\mathrm{Dep}(p)}^{\otimes 5}, 2\right)$ as an upper bound on $p\left(\mathcal{N}_{\mathrm{Dep}(p)}^{\otimes 5}, 2\right)$ in MATLAB using CVX and MOSEK:





[Reimpell & Werner '05]

- For $p \in [1, 4/3]$ the codes from [Reimpell & Werner '05] are optimal — for $p \in [0, 0.18]$ there is room to look for improved codes.

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Conclusion

Take home message:

Methods form optimization theory for quantifying the difference between classical and quantum problems in information processing.

- Relevant bilinear programs in general hard but approximated via non-commutative sum-of-squares hierarchies.
- Analytical question: Identify operational settings in quantum information that allow for efficient approximation. For example, free games / quantum adversaries in classical cryptography / classical-quantum channel coding?
- Numerical question: Dimension reduction tools exploiting symmetries needed (group theory) $\Rightarrow$ extensive numerics for quantum channel coding practically relevant for NISQ technology.
- Multilinear / multipartite extensions?

Thanks!

[SIAM Journal on Optimization '16] with Fawzi and Scholz
[arXiv:1810.12197] with Borderi, Fawzi, and Scholz