Quantum Computing CO484

Tutorial*

Sheet 4 – Questions

Exercise 1 *Consider Grover's algorithm (as in the lecture notes).*

(i) Show that

$$(2|\psi\rangle\langle\psi| - \mathbf{I})(\sum_x \alpha_x |x\rangle) = \sum_x (-\alpha_x + 2\langle\alpha\rangle)|x\rangle$$

where $\langle\alpha\rangle = \frac{1}{N}\sum_x \alpha_x$.

(ii) Explain why the operation $2|\psi\rangle\langle\psi| - \mathbf{I}$ *in Grover's algorithm is called* inversion about mean.

Exercise 2 *Show that, in Grover's algorithm, the action of* $2|\psi\rangle\langle\psi| - \mathbf{I}$ *is a reflection about* $|\psi\rangle$.

Exercise 3 *Show that if we change the computational basis so that* $|\sigma\rangle$ *and* $|\tau\rangle$ *are basis elements, then the matrix representation of the Grover's transform* G *will be;*

$$G_{\sigma,\tau} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\theta/2$ *is the angle between* $|\sigma\rangle$ *and* $|\psi\rangle$.

Exercise 4 ** We have shown in the lecture that a quantum computer can search* N *items, consulting the black box that can access the database and tells you if an item is marked only* $O(\sqrt{N})$ *times. We now prove that no quantum algorithm can perform this task using fewer than* $\Omega(\sqrt{N})$ *accesses to the black box. Suppose the algorithm starts in the state* $|\phi\rangle$. *To determine*

*Partly based on the tutorials by Abbas Edalat and Herbert Wiklicky.

the marked x we are allowed to apply a black box Q_x, which gives a phase shift of -1 to the solution $|x\rangle$ and leaves all other states invariant, i.e.,

$$Q_x = 1 - 2|x\rangle\langle x|.$$

We suppose the algorithm applies Q_x exactly k-times, with unitary operations $U_1, U_2, \ldots, U_k$ interleaved between the black box operations. Define

$$|\phi_k^x\rangle = U_k Q_x U_{k-1} Q_x \ldots U_1 Q_x |\phi\rangle, \quad |\phi_k\rangle = U_k U_{k-1} \ldots U_1 |\phi\rangle,$$

and let $|\phi_0\rangle = |\phi\rangle$. Our goal is then to bound the quantity

$$D_k = \sum_x \|\phi_k^x - \phi_k\|^2 .$$

(i) *Show that $D_k \leq 4k^2$.*
Hint: *Do a proof by induction and employ the Cauchy-Schwarz inequality.*

(ii) *Show that if the algorithm yields a solution with probability at least one-half, i.e., $|\langle x|\phi_k^x\rangle|^2 \geq 1/2$ for all x, then $D_k = \Omega(N)$.*
Hint: *Employ the Cauchy-Schwarz inequality.*

(iii) *Argue why this shows that Grover's algorithm is optimal.*

(iv) *Discuss why this result is both exciting and disappointing.*