Quantum Computation (484)

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Indicator Function

We will search for a (single) element $\mathbf{x}_0$ which is identified via an indicator function $f : \{0,1\}^n \to \{0,1\}$:

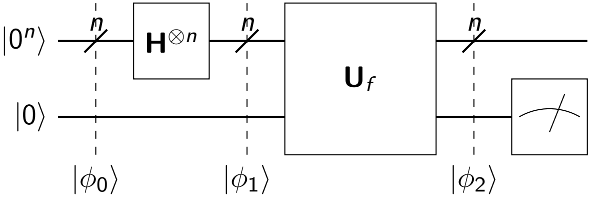
$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{x}_0 \\ 0 & \text{otherwise} \end{cases}$$

As usual, we represent this function as a unitary $\mathbf{U}_f$ which maps $|\mathbf{x}, y\rangle$ to $|\mathbf{x}, f(\mathbf{x}) \oplus y\rangle$. For example: to pick the element "10" out of $\{00, 01, 10, 11\}$ we use

$$\mathbf{U}_f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Superposition

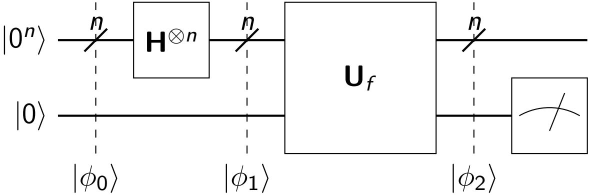
A first attempt could try to evaluate f or $\mathbf{U}_f$ on all possible inputs.



$$|\phi_0\rangle = |0^n, 0\rangle$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2^n}} \left(\sum_{\mathbf{x} \in \{0,1\}^n} |\mathbf{x}\rangle \right) |0\rangle$$

Measuring the Final State

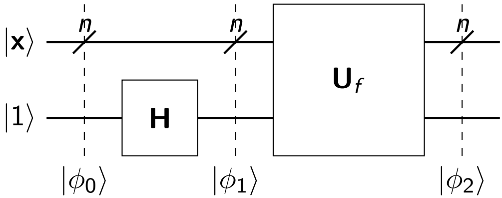


We can measure the top n qubits or the lower qubit of the final state:

$$|\phi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x} \in \{0,1\}^n} |\mathbf{x}, f(\mathbf{x})\rangle$$

If we measure the bottom qubit and if we are **lucky** we might measure $|1\rangle$. In this case the top qubits will collapse to the desired outcome $\mathbf{x}_0$. Chances for this are just $\frac{1}{2^n}$.

Phase Inversion

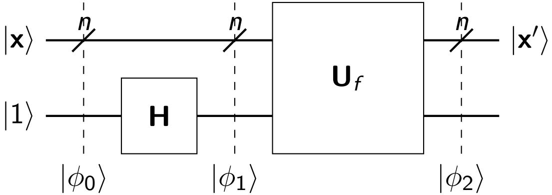


$$|\phi_0\rangle = |\mathbf{x}, 1\rangle$$

$$|\phi_1\rangle = |\mathbf{x}\rangle \left(\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right) = \frac{1}{\sqrt{2}} (|\mathbf{x}, 0\rangle - |\mathbf{x}, 1\rangle)$$

$$|\phi_2\rangle = |\mathbf{x}\rangle \frac{1}{\sqrt{2}} (|f(\mathbf{x}) \oplus 0\rangle - |f(\mathbf{x}) \oplus 1\rangle) = |\mathbf{x}\rangle \frac{1}{\sqrt{2}} \left(|f(\mathbf{x})\rangle - |\overline{f(\mathbf{x})}\rangle \right)$$

Inverted State



$$|\phi_2\rangle = (-1)^{f(\mathbf{x})} |\mathbf{x}\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \begin{cases} -\frac{1}{\sqrt{2}} |\mathbf{x}\rangle (|0\rangle - |1\rangle) & \text{if } \mathbf{x} = \mathbf{x}_0 \\ +\frac{1}{\sqrt{2}} |\mathbf{x}\rangle (|0\rangle - |1\rangle) & \text{if } \mathbf{x} \neq \mathbf{x}_0 \end{cases}$$

Example: Searching for "10" in $\{00, 01, 10, 11\}$. Starting with a uniform state $|\mathbf{x}\rangle = \frac{1}{2}(1, 1, 1, 1)^T$ as top qubits, these become $|\mathbf{x}'\rangle = \frac{1}{2}(1, 1, -1, 1)^T$.

However, measuring $|\mathbf{x}'\rangle$ does not identify $\mathbf{x}_0$.

Inversion about the Mean

Consider, for example, a sequence of numbers $53, 38, 17, 23, 79$.

The **average** or **mean** of these is

$$a = 42 = \frac{53 + 38 + 17 + 23 + 79}{5}.$$

Replace every number v by its mirror image through a, i.e.

$$\begin{aligned} v' &= a + (a - v) \\ &= -v + 2a \end{aligned}$$

In our example we get the new sequence $31, 46, 67, 61, 5$ which has the same average $a = 42$.

Average Operator

Represent sequences as vectors $\mathbf{v}$, e.g. $\mathbf{v} = (53, 38, 17, 23, 79)^T$.

We can introduce an averaging operator for *n*-dimensional vectors

$$(\mathbf{A})_{ij} = \frac{1}{n} \quad \text{e.g. } \mathbf{A} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

With this we can compute the average a of a sequence $\mathbf{v}$ by:

$$\mathbf{Av} = (a, a, \ldots, a)^T$$

In our example, $\mathbf{A}(53, 38, 17, 23, 79)^T = (42, 42, 42, 42, 42)^T$.

Mean Inversion Operator

We can also define a matrix representation of the inversion about the mean or average, which is in fact **unitary** (show!):

$$\begin{aligned} \mathbf{v}' &= -\mathbf{v} + 2\mathbf{A}\mathbf{v} \\ &= (-\mathbf{I} + 2\mathbf{A})\mathbf{v} \end{aligned}$$

In our example the mean inversion operator is:

$$2\mathbf{A} - \mathbf{I} = \begin{pmatrix} \frac{2}{5} - 1 & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} - 1 & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} - 1 & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} - 1 & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} - 1 \end{pmatrix}$$

Thus we get: $(-\mathbf{I} + 2\mathbf{A})(53, 38, 17, 23, 79)^T = (31, 46, 67, 61, 5)^T$.

Amplitude Amplifying

Consider the following uniform vector of values:

$$\mathbf{v}_0 = (10, 10, 10, 10, 10)^T$$

Perform a **phase inversion** on the 4th element. The result is:

$$\mathbf{v}_1 = (10, 10, 10, -10, 10)^T$$

with average $a = 6$. Perform an **inversion about the mean** to get

$$\mathbf{v}_2 = (2, 2, 2, 22, 2)^T$$

followed by another **phase inversion**

$$\mathbf{v}_3 = (2, 2, 2, -22, 2)^T$$

and an **inversion about the mean** $a = -2.8$ to get:

$$\mathbf{v}_4 = (-7.6, -7.6, -7.6, 16.4, -7.6)^T$$

Grover's Algorithm

Use phase inversion and inversion about the mean to make it more likely to be **lucky**. These amplifications need to be applied $\sqrt{2^n}$ times for an n qubit register (i.e. 2^n search items).

Step 1. Start with $|0^n\rangle$.

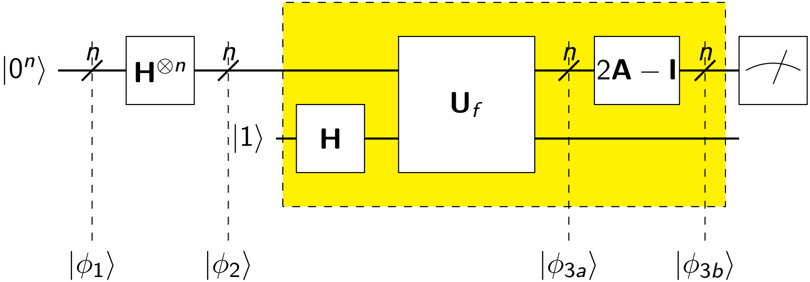
Step 2. Apply $\mathbf{H}^{\otimes n}$.

Step 3. Repeat $\sqrt{2^n}$ times:

- Step 3a. **Phase inversion** $\mathbf{U}_f(\mathbf{I} \otimes \mathbf{H})$.
- Step 3b. **Inversion about the mean** $-\mathbf{I} + 2\mathbf{A}$.

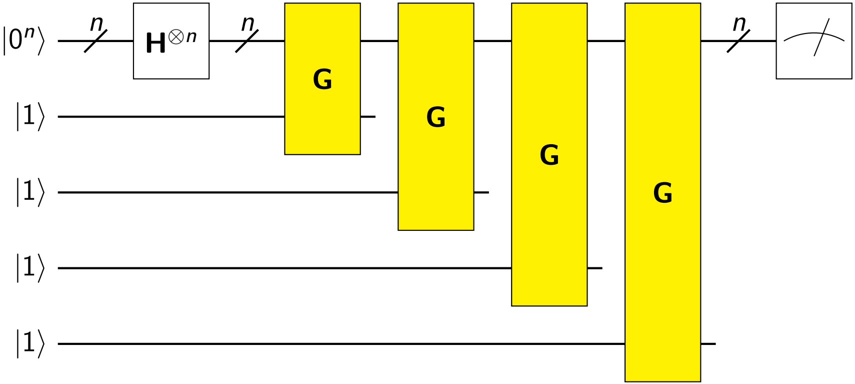
Step 4. Measurement.

Grover's Circuit

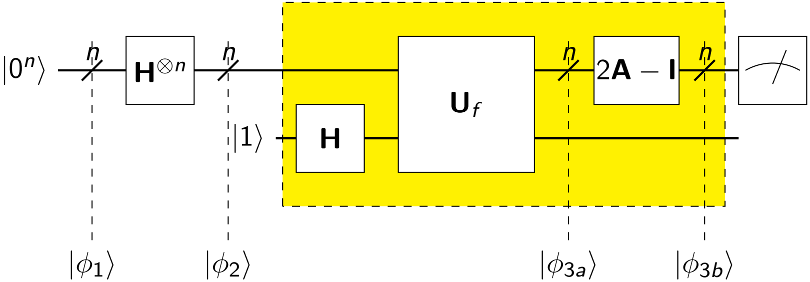


Classical search in an unordered array of size m needs m steps in the worst case and $\frac{m}{2}$ in the average. Grover's search algorithm will take $\sqrt{m}$ steps.

Iterations of Grover's Gate



Example: Preparation



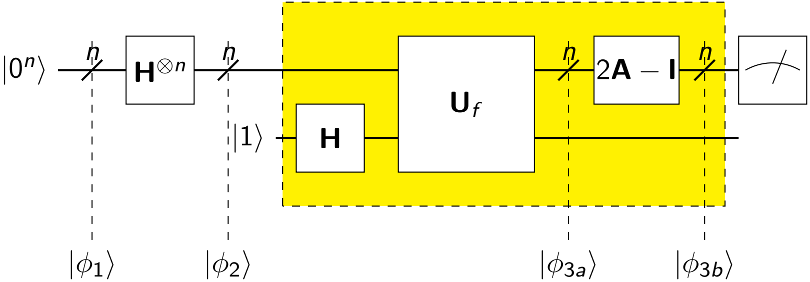
Find "110" in $\{000, 001, 010, 011, 100, 101, 110, 111\}$. We need $\sqrt{8} = 2.8284$ iterations.

Preparation Step.

$$|\phi_1\rangle = (1, 0, 0, 0, 0, 0, 0, 0)^T \otimes \ldots$$

$$|\phi_2\rangle = \frac{1}{\sqrt{8}}(1, 1, 1, 1, 1, 1, 1, 1)^T \otimes \ldots$$

Example: 1st Iteration



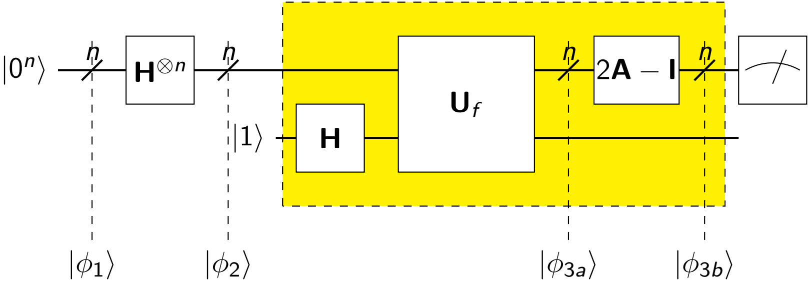
1st Iteration. Phase inversion:

$$|\phi_{3a}\rangle = \frac{1}{\sqrt{8}}(1,1,1,1,1,1,-1,1)^T \otimes \dots$$

Inversion about the mean $a = \frac{1}{\sqrt{8}}\frac{3}{4}$:

$$|\phi_{3b}\rangle = \frac{1}{\sqrt{8}}(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{1}{2})^T \otimes \dots$$

Example: 2nd Iteration



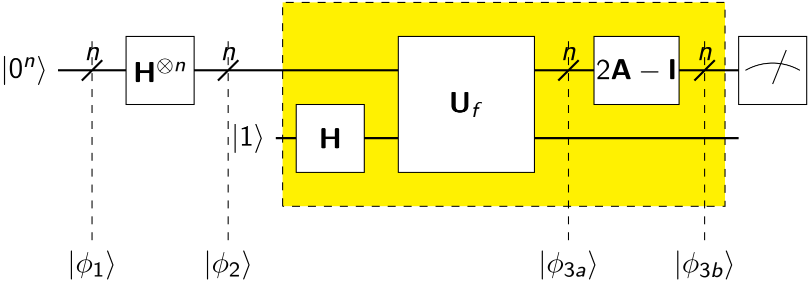
2nd Iteration. Phase inversion:

$$|\phi_{3a}\rangle = \frac{1}{\sqrt{8}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{5}{2}, \frac{1}{2})^T \otimes \dots$$

Inversion about the mean $a = \frac{1}{\sqrt{8}}\frac{1}{8}$:

$$|\phi_{3b}\rangle = \frac{1}{\sqrt{8}}(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{11}{4}, -\frac{1}{4})^T \otimes \dots$$

Example: Measurement



Measuring the final state

$$|\phi_{3b}\rangle = \frac{1}{\sqrt{8}}(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{11}{4}, -\frac{1}{4})^T \otimes \dots$$

in the standard base $|000\rangle, |001\rangle, \dots$. Will return (collapses to) one of the base vectors. We get with probability $|\frac{1}{\sqrt{8}}\frac{11}{4}|^2 = 0.97227^2 = 0.94531$:

$$|110\rangle = (0, 0, 0, 0, 0, 0, 1, 0)^T$$

Alternative Formulation: Oracle

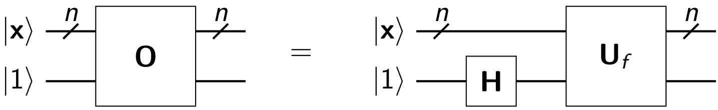
Some texts start immediately with a 'uniform' initial vector

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{x=0}^{n-1} |x\rangle$$

and use an **oracle O** for picking the element we look for.

$$\mathbf{O} : \sum_{\mathbf{x} \in \{0,1\}^n} \alpha_{\mathbf{x}} |\mathbf{x}\rangle \mapsto \sum_{\mathbf{x} \in \{0,1\}^n} (-1)^{f(\mathbf{x})} \alpha_{\mathbf{x}} |\mathbf{x}\rangle$$

In our presentation we obtained $|\psi\rangle$ explicitly by applying $\mathbf{H}^{\otimes n}$ while the oracle corresponds to the phase inversion:



Realisation of the Inversion about the Mean

As it is a unitary operator we know that $2\mathbf{A} - \mathbf{I}$ can be represented as a quantum gate. Concretely, some authors use the **conditional phase shift $\mathbf{P}_0$** to implement it.

$$\mathbf{P}_0 : |\mathbf{x}\rangle \mapsto \begin{cases} |\mathbf{x}\rangle & \text{if } \mathbf{x} = 0^n \\ -|\mathbf{x}\rangle & \text{otherwise} \end{cases}$$

As we can write $\mathbf{P}_0 = 2\,|0^n\rangle\langle 0^n| - \mathbf{I}$ we obtain

$$\mathbf{H}^{\otimes n}\mathbf{P}_0\mathbf{H}^{\otimes n} = \mathbf{H}^{\otimes n}(2\,|0^n\rangle\langle 0^n| - \mathbf{I})\mathbf{H}^{\otimes n} = 2\,|\psi\rangle\langle\psi| - \mathbf{I} = 2\mathbf{A} - \mathbf{I}$$

The Grover operator $\mathbf{G} = ((2\mathbf{A} - \mathbf{I}) \otimes \mathbf{I})\mathbf{O}$ can be represented by