

Matrix Trace Inequalities for Quantum Entropy

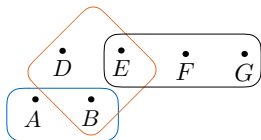
Mario Berta

based on joint work with Brandão, Fawzi, Hirche, Sutter, Tomamichel

Motivation

- **Entropy** of quantum states ρ_A on Hilbert spaces \mathcal{H}_A [von Neumann 27]:

$$H(A)_\rho := -\text{tr} [\rho_A \log \rho_A]$$



Entropy for multipartite quantum systems:
entanglement leads to non-commutativity

- **Mathematical properties** of quantum entropy from:

Strong subadditivity (SSA) [Lieb, Ruskai 73]

For tripartite quantum states ρ_{ABC} on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ we have

$$H(AB)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(B)_\rho$$

- SSA is an essential tool in all of quantum information science since it gives constraints on **multipartite entanglement structure**

Motivation

- Understanding quantum entropy goes hand in hand with **matrix trace inequalities**, e.g., for H_1, H_2 Hermitian

$$\exp(H_1) \exp(H_2) = \exp\left(H_1 + H_2 + \frac{1}{2}[H_1, H_2] + \dots\right) \quad \text{Baker-Campbell-Hausdorff}$$

$$\text{tr} [\exp(H_1) \exp(H_2)] \geq \text{tr} [\exp(H_1 + H_2)] \quad \text{Golden-Thompson}$$

- Entropy for multipartite quantum systems requires **multivariate trace inequalities**:

SSA from Lieb's triple matrix inequality [Lieb 73]

For H_1, H_2, H_3 Hermitian we have

$$\int_0^\infty \text{tr} \left[\exp(H_1) \frac{1}{\exp(-H_3) + \lambda} \exp(H_2) \frac{1}{\exp(-H_3) + \lambda} \right] d\lambda \geq \text{tr} [\exp(H_1 + H_2 + H_3)]$$

$$\left(\text{commutative case: } \int_0^\infty \frac{1}{(x^{-1} + \lambda)^2} d\lambda = x \right)$$

Overview

- 1 From classical to quantum entropy
- 2 Entropy inequalities
- 3 Multivariate trace inequalities
- 4 Proofs via complex interpolation theory
- 5 Proof of entropy inequalities
- 6 Outlook

Entropy for classical systems

- **Entropy** of probability distribution P of random variable X over finite alphabet [Shannon 48, Rényi 61]:

$$H(X)_P := - \sum_x P(x) \log P(x) \quad \text{with } P(x) \log P(x) = 0 \text{ for } P(x) = 0$$

- **Relative entropy** of P with respect to distribution Q over finite alphabet,

$$D(P\|Q) := \sum_x P(x) \log \frac{P(x)}{Q(x)} \quad [\text{Kullback, Leibler 51}]$$

where $P(x) \log \frac{P(x)}{Q(x)} = 0$ for $P(x) = 0$ and by continuity $+\infty$ if $P \not\ll Q$.

- **Multipartite entropy measures** are generated through relative entropy, e.g., SSA:

$$H(XY)_P + H(YZ)_P \geq H(XYZ)_P + H(Y)_P \quad \text{equivalent to}$$

$$D(P_{XYZ}\|U_X \times P_{YZ}) \geq D(P_{XY}\|U_X \times P_Y) \quad \text{with } U_X \text{ uniform distribution}$$

- **Monotonicity of relative entropy (MONO)** under stochastic matrices N :

$$D(P\|Q) \geq D(N(P)\|N(Q))$$

Entropy for quantum systems

- The **entropy** of $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ is defined as:

$$H(A)_\rho := -\text{tr}[\rho_A \log \rho_A] = -\sum_x \lambda_x \log \lambda_x \quad [\text{von Neumann 27}]$$

- Commutative relative entropy** for $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ defined as

$$D_K(\rho\|\sigma) := \sup_{\mathcal{M}} D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)) \quad [\text{Donald 86}]$$

$\mathcal{M} \in Q_{OP}(\mathcal{H} \rightarrow \mathcal{H}')$ und $\text{Im}(\mathcal{M}) \subseteq M \subseteq \text{Lin}(\mathcal{H}')$, M commutative subalgebra.

- Some alternative **non-commutative extensions** of $D(P\|Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)}$:

$$D(\rho\|\sigma) := \text{tr}[\rho(\log \rho - \log \sigma)] \quad [\text{Umegaki 62}]$$

$$D_B(\rho\|\sigma) := \text{tr}\left[\rho \log\left(\rho^{1/2} \sigma^{-1} \rho^{1/2}\right)\right] \quad [\text{Belavkin, Staszewski 82}]$$

- Monotonicity of relative entropy (MONO)** for $\mathcal{N} \in Q_{OP}(\mathcal{H} \rightarrow \mathcal{H}')$:

$$D_K(\rho\|\sigma) \geq D_K(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \quad [\text{Lindblad 75}]$$

$$D_B(\rho\|\sigma) \geq D_B(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \quad [\text{Fujii, Kamei 89}]$$

Entropy for quantum systems

Quantum relative entropy

For $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with $\rho, \sigma > 0$ we have

$$D_K(\rho\|\sigma) \leq D(\rho\|\sigma) \leq D_B(\rho\|\sigma) \quad \text{with equality if and only if } [\rho, \sigma] \neq 0.$$

- Access through **variational formulas** (Legendre transformations):

$$D_K(\rho\|\sigma) = \sup_{\omega > 0} \text{tr} [\rho \log \omega] - \log \text{tr} [\sigma \omega] \quad [\text{Donald 86}], [\text{B.}, \text{Fawzi}, \text{Tomamichel 17}]$$

$$D(\rho\|\sigma) = \sup_{\omega > 0} \text{tr} [\rho \log \omega] - \log \text{tr} [\exp(\log \sigma + \log \omega)] \quad [\text{Araki (first)}], [\text{Petz 88}]$$

$$D_B(\rho\|\sigma) = \text{tr} [\rho \log \omega] - \log \text{tr} [\sigma \# \omega] \quad \text{for } \omega := \rho^{1/2} \sigma^{-1} \rho^{1/2} \quad [\text{Ando, Hiai 94}]$$

with $\sigma \# \omega := \sigma^{1/2} (\sigma^{-1/2} \omega \sigma^{-1/2})^{1/2} \sigma^{1/2}$ the matrix geometric mean.

- **Golden-Thompson** and complementary inequality

$$\text{tr}[\exp(M_1) \exp(M_2)] \geq \text{tr}[\exp(M_1 + M_2)] \geq \text{tr}[\exp(M_1) \# \exp(M_2)]$$

Entropy for quantum systems

- The right extension for applications is Umegaki's $D(\rho\|\sigma)$, intuition comes from the **chain rule** [Petz 92] with SSA:

$$H(AB)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(B)_\rho \quad \text{equivalent to}$$

$$D(\rho_{ABC}\|\tau_A \otimes \rho_{BC}) \geq D(\rho_{AB}\|\tau_A \otimes \rho_B) \quad \text{with } \tau_A = \frac{1_A}{\dim(\mathcal{H}_A)}.$$

- All known mathematical properties from MONO $D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$:

Equality conditions MONO [Petz 86]

Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with $\rho \ll \sigma$ and $\mathcal{N} \in Q_{OP}(\mathcal{H} \rightarrow \mathcal{H}')$. Then, we have

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = 0$$

if and only if there exists $\mathcal{R}_{\sigma, \mathcal{N}} \in Q_{OP}(\mathcal{H}' \rightarrow \mathcal{H})$ such that

$$\mathcal{R}_{\sigma, \mathcal{N}} \circ \mathcal{N}(\rho) = \rho \quad \text{und} \quad \mathcal{R}_{\sigma, \mathcal{N}} \circ \mathcal{N}(\sigma) = \sigma.$$

The quantum operation $\mathcal{R}_{\sigma, \mathcal{N}}$ is not unique but can be chosen independent of ρ .

Strong monotonicity (sMONO) [many recent references]

For the same premises we have $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq (1), (2), (3)$ for

$$(1) := \int \beta_0(t) - \log F\left(\rho, \mathcal{R}_{\sigma, \mathcal{N}}^{[t]}(\rho)\right) dt \quad [\text{Junge, Renner, Sutter, Wilde, Winter 15}]$$

$$(2) := D_K\left(\rho \parallel \int \beta_0(t) \mathcal{R}_{\sigma, \mathcal{N}}^{[t]}(\rho) dt\right) \quad [\text{Sutter, B., Tomamichel 16}]$$

$$(3) := \limsup_{n \rightarrow \infty} \frac{1}{n} D\left(\rho^{\otimes n} \parallel \int \beta_0(t) \left(\mathcal{R}_{\sigma, \mathcal{N}}^{[t]}(\rho)\right)^{\otimes n} dt\right) \quad [\text{B., Brand\~{a}o, Hirche 17}]$$

with the probability distribution $\beta_0(t) := \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}$ and

$$\mathcal{R}_{\sigma, \mathcal{N}}^{[t]}(\cdot) := \sigma^{\frac{1+it}{2}} \mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-\frac{1+it}{2}}(\cdot)\mathcal{N}(\sigma)^{-\frac{1-it}{2}}\right) \sigma^{\frac{1-it}{2}} \in Q_{OP}(\mathcal{H}' \rightarrow \mathcal{H})$$

- We have $D(\rho\|\sigma) \geq D_K(\rho\|\sigma) \geq -\log F(\rho, \sigma)$ but the regularization in (3) is needed [Fawzi² 17], leading to incomparable bounds [B., Brand\~{a}o, Hirche 17].

Entropy inequalities

- Special case of quantum Markov chains $A - B - C$:

Strong SSA (sSSA) [same references]

For $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ we have for the quantum conditional mutual information (CQMI)

$$\begin{aligned} I(A : C|B)_\rho &:= H(AB)_\rho + H(BC)_\rho - H(ABC)_\rho - H(B)_\rho \\ &\geq D_K \left(\rho_{ABC} \left\| \int \beta_0(t) \left(\mathcal{I}_A \otimes \mathcal{R}_{B \rightarrow BC}^{[t]} \right) (\rho_{AB}) dt \right. \right) \end{aligned}$$

with

$$\mathcal{R}_{B \rightarrow BC}^{[t]}(\cdot) := \rho_{BC}^{\frac{1+it}{2}} \left(\left(\rho_B^{-\frac{1+it}{2}}(\cdot)\rho_B^{-\frac{1-it}{2}} \right) \otimes \mathbf{1}_C \right) \rho_{BC}^{\frac{1-it}{2}} \in \mathcal{Q}_{OP}(\mathcal{H}_B \rightarrow \mathcal{H}_B \otimes \mathcal{H}_C).$$

(Moreover, we have the analogue fidelity and quantum relative entropy bounds.)

- Proofs of weaker versions of sSSA [Fawzi, Renner 15] + [many references] \Rightarrow matrix analysis will give tight bounds.

Entropy inequalities

- **Proof SSA:** Following [Lieb, Ruskai 73] we have with Klein's inequality

$$\begin{aligned} I(A : C|B)_\rho &= D(\rho_{ABC} \| \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})) \\ &\geq \text{tr} [\rho_{ABC} - \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})] \end{aligned}$$

and we could conclude SSA if $\text{tr}[\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})] \leq 1$. However, just take Lieb's triple matrix inequality

$$\text{tr} [\exp(\log M_1 + \log M_2 + \log M_3)] \leq \int_0^\infty \text{tr} \left[M_1 \frac{1}{M_2^{-1} + \lambda} M_3 \frac{1}{M_2^{-1} + \lambda} \right] d\lambda$$

with $M_1 := \rho_{AB}$, $M_2 := \rho_B^{-1}$, $M_3 := \rho_{BC}$, and use $\int_0^\infty d\lambda x(x^{-1} + \lambda)^{-2} = 1$. \square

- **Proof sSSA:** Start with the variational formula

$$\begin{aligned} I(A : C|B)_\rho &= D(\rho_{ABC} \| \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})) \\ &= \sup_{\omega_{ABC} > 0} \text{tr} [\rho_{ABC} \log \omega_{ABC}] - \log \text{tr} [\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC} + \log \omega_{ABC})] \end{aligned}$$

but now **four matrix extension** of Golden-Thompson inequality needed?

Multivariate trace inequalities

- Golden-Thompson $\|\exp(H_1 + H_2)\| \leq \|\exp(H_1) \exp(H_2)\|$ to multivariate:

[Sutter, B., Tomamichel 16], [Hiai, Koenig, Tomamichel 17]

For $\{H_k\}_{k=1}^n$ Hermitian and $\beta_0(t) := \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}$ we have

$$\log \left\| \exp \left(\sum_{k=1}^n H_k \right) \right\| \leq \int \beta_0(t) \log \left\| \prod_{k=1}^n \exp((1 + it)H_k) \right\| dt$$

for any unitarily invariant norm $\|\cdot\|$.

- For $n = 3$, $\|\cdot\| = \|\cdot\|_2$ this relaxes to (Jensen's inequality)

$$\begin{aligned} & \operatorname{tr} [\exp(H_1 + H_2 + H_3)] \\ & \leq \int \beta_0(t) \operatorname{tr} \left[\exp(H_1) \exp\left(\frac{1 + it}{2} H_2\right) \exp(H_3) \exp\left(\frac{1 - it}{2} H_2\right) \right] dt \end{aligned}$$

which becomes Lieb's bound $\int_0^\infty \operatorname{tr} \left[\exp(H_1) \frac{1}{\exp(-H_2) + \lambda} \exp(H_3) \frac{1}{\exp(-H_2) + \lambda} \right] d\lambda$.

Multivariate trace inequalities

- Lie-Trotter expansion $\exp\left(\sum_{k=1}^n \log P_k\right) = \lim_{r \rightarrow 0} \left|\prod_{k=1}^n P_k^r\right|^{1/r}$:

Multivariate Araki-Lieb-Thirring [same references]

For $\{M_k\}_{k=1}^n$ positive, $r \in (0, 1]$, and $\beta_r(t) := \frac{\sin(\pi r)}{2r(\cosh(\pi t) + \cos(\pi r))}$ we have

$$\log \left\| \left\| \prod_{k=1}^n M_k^r \right\|^{1/r} \right\| \leq \int \beta_r(t) \log \left\| \left\| \prod_{k=1}^n M_k^{1+it} \right\| \right\| dt$$

- L'Hôpital's rule yields another multivariate extension:

Multivariate Logarithmic Hiai-Petz [Sutter, B., Tomamichel 17]

For $\{M_k\}_{k=1}^n$ positive and $p > 0$ we have

$$\sum_{k=1}^n \operatorname{tr} [M_k \log M_k] \geq \frac{1}{p} \int \beta_0(t) \operatorname{tr} \left[M_1^{\frac{p(1+it)}{2}} \cdots M_{n-1}^{\frac{p}{2}} M_n^p M_{n-1}^{\frac{p}{2}} \cdots M_1^{\frac{p(1-it)}{2}} \right] dt$$

with equality in the limit $p \rightarrow 0$.

Proofs via complex interpolation theory

- Matrix analysis: [Epstein 73], [Kosaki 97]. QIT: [Beigi 13], [Dupuis 15], [Wilde 15], [Junge, Renner, Sutter, Wilde, Winter 15], [Dupuis, Wilde 16].

Strong Hadamard three line theorem [Hirschman 53]

Let $S := \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq 1\}$, $g : S \rightarrow \mathbb{C}$ be uniformly bounded on S , holomorphic in the interior of S , and continuous on the boundary. Then, we have for $r \in (0, 1)$ with

$\beta_r(t) := \frac{\sin(\pi r)}{2r(\cosh(\pi t) + \cos(\pi r))}$ that:

$$\begin{aligned} \log |g(r)| &\leq \int \left(\beta_{1-r}(t) \log |g(it)|^{1-r} + \beta_r(t) \log |g(1+it)|^r \right) dt \\ &\leq \sup_t \log |g(it)|^{1-r} + \sup_t \log |g(1+it)|^r \end{aligned}$$

- For $G(z)$ matrix valued uniformly bounded holomorphic function we have [Stein]:

$$\log \|G(r)\|_\infty \leq \int dt \left(\beta_{1-r}(t) \log \|G(it)\|_\infty^{1-r} + \beta_r(t) \log \|G(1+it)\|_\infty^r \right)$$

- Proof: For $r \in (0, 1)$ let $u, v \in \mathbb{C}^d$ be normalized s.t. $\|G(r)\|_\infty = \langle u, G(r)v \rangle$ and hence for $g(z) := \langle u, G(z)v \rangle$ we get $|g(z)| \leq \|G(z)\|_\infty$ for all $z \in S$. \square

Proofs via complex interpolation theory

- **Multivariate Araki-Lieb-Thirring** for $\|\cdot\| = \|\cdot\|_\infty$ with $r \in (0, 1]$ and $M_k > 0$:

$$\log \left\| \left\| \prod_{k=1}^n M_k^r \right\|^{1/r} \right\|_\infty \leq \int \beta_r(t) \log \left\| \prod_{k=1}^n M_k^{1+it} \right\|_\infty dt$$

- Proof: Apply Stein-Hirschman

$$\log \|G(r)\|_\infty \leq \int dt \left(\beta_{1-r}(t) \log \|G(it)\|_\infty^{1-r} + \beta_r(t) \log \|G(1+it)\|_\infty^r \right)$$

for the function

$$G(z) := \prod_{k=1}^n M_k^z = \prod_{k=1}^n \exp(z \log M_k)$$

and since $M_k > 0$, M_k^{it} becomes unitary, and hence $\log \|\cdot\|_\infty^{1-r} = 0$. \square

- General case for unitarily invariant norms via anti-symmetric tensor product (or for Schatten p -norms with more general Stein-Hirschman interpolation).
- **Multivariate Golden-Thompson** from Lie-Trotter expansion.

Proof of entropy inequalities

- For the proof of sSSA we choose $n = 4$, $\|\cdot\| = \|\cdot\|_2$ and we get for $M_1, M_2, M_3, M_4 > 0$ (Lieb's triple matrix inequality corresponds to $n = 3$):

$$\begin{aligned} & \text{tr} [\exp(\log M_1 + \log M_2 + \log M_3 + \log M_4)] \\ & \leq \int \beta_0(t) \text{tr} \left[M_1 M_2^{\frac{1+i}{2}} M_3^{\frac{1+i}{2}} M_4 M_3^{\frac{1-i}{2}} M_2^{\frac{1-i}{2}} \right] dt \end{aligned}$$

- Choose $M_1 := \omega_{ABC}$, $M_2 := \rho_{BC}$, $M_3 := \rho_B^{-1}$, $M_4 := \rho_{AB}$ and thus

$$\begin{aligned} I(A : C|B)_\rho &= D(\rho_{ABC} \| \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})) \\ &= \sup_{\omega_{ABC} > 0} \text{tr} [\rho_{ABC} \log \omega_{ABC}] - \log \text{tr} [\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC} + \log \omega_{ABC})] \\ &\geq \sup_{\omega_{ABC} > 0} \text{tr} [\rho_{ABC} \log \omega_{ABC}] - \int \beta_0(t) \log \text{tr} \left[\omega_{ABC} \rho_{BC}^{\frac{1+i}{2}} \rho_B^{-\frac{1+i}{2}} \rho_{AB} \rho_B^{-\frac{1-i}{2}} \rho_{BC}^{\frac{1-i}{2}} \right] dt \\ &\geq D_K \left(\rho_{ABC} \| \int dt \beta_0(t) \rho_{BC}^{\frac{1+i}{2}} \rho_B^{-\frac{1+i}{2}} \rho_{AB} \rho_B^{-\frac{1-i}{2}} \rho_{BC}^{\frac{1-i}{2}} \right) \quad \square \end{aligned}$$

- All sSSA/sMONO bounds follow from multivariate Golden-Thompson.

Outlook

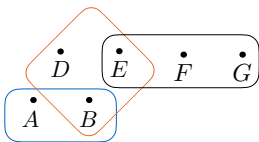
- Strengthened entropy inequalities sSSA/sMONO through multivariate matrix trace inequalities from complex interpolation theory
- Different representation of integrals [Lemm 17], [Garg, Lee, Song, Srivastava 17]:

$$\int \beta_0(t) \cdots dt \quad \text{vs.} \quad \int_0^\infty \cdots d\lambda \quad \text{vs.} \quad \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \cdots d\mu(\phi)$$

- Recent development in MathPhys, e.g., [Carlen, Lieb 17], [Carlen, Vershynina 17]
- **Multivariate matrix analysis?** Complementary trace inequalities [Hiai, Petz 08]:

$$\text{tr}[\exp(M_1) \# \exp(M_2)] \leq \text{tr}[\exp(M_1 + M_2)] \leq \text{tr}[\exp(M_1) \exp(M_2)]$$

- **Novel quantum entropy inequalities?** For example non-Shannon type information inequalities for the non-commutative entropy cone [Pippenger 03].



- Quantum information science: novel **constraints on multipartite entanglement structure** [very many references]