Converse bounds for private communication over quantum channels

Mario Berta

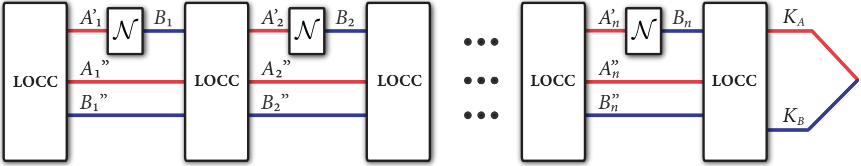
joint work with Mark M. Wilde and Marco Tomamichel, arXiv:1602.08898

TQC Berlin - September 27, 2016

Caltech

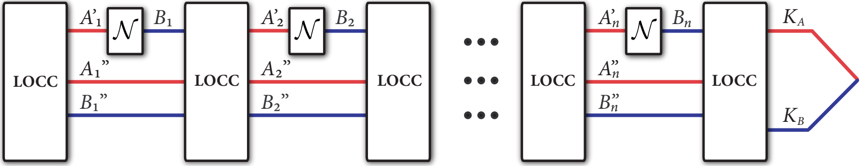
Setup I

- Given a quantum channel $\mathcal{N}$ and a quantum key distribution (**QKD**) protocol that uses it n times, how much **key** can be generated?



Setup I

- Given a quantum channel $\mathcal{N}$ and a quantum key distribution (**QKD**) protocol that uses it n times, how much **key** can be generated?



- **Non-asymptotic private capacity**: maximum rate of ε-close secret key achievable using the channel n times with two-way classical communication (LOCC) assistance

$$\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon) := \sup\left\{P : (n,P,\varepsilon) \text{ is achievable for } \mathcal{N} \text{ using LOCC}\right\}. \quad (1)$$

Setup II

- Practical question: how to characterize $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon)$ for all $n \geq 1$ and $\varepsilon \in (0,1)$? The answers give the **fundamental limitations of QKD**.

Setup II

- Practical question: how to characterize $\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon)$ for all $n \geq 1$ and $\varepsilon \in (0,1)$? The answers give the **fundamental limitations of QKD**.
- Upper bounds on $\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon)$ can be used as **benchmarks for quantum repeaters** [Lütkenhaus].

Setup II

- Practical question: how to characterize $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon)$ for all $n \geq 1$ and $\varepsilon \in (0,1)$? The answers give the **fundamental limitations of QKD**.
- Upper bounds on $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon)$ can be used as **benchmarks for quantum repeaters** [Lütkenhaus].
- Today, I will present

  the tightest known upper bound on $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon)$

  for many channels of practical interest. Interesting special case: single-mode phase-insensitive bosonic Gaussian channels.

Setup II

- Practical question: how to characterize $\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon)$ for all $n \geq 1$ and $\varepsilon \in (0,1)$? The answers give the **fundamental limitations of QKD**.
- Upper bounds on $\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon)$ can be used as **benchmarks for quantum repeaters** [Lütkenhaus].
- Today, I will present

the tightest known upper bound on $\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon)$

  for many channels of practical interest. Interesting special case: single-mode phase-insensitive bosonic Gaussian channels.
- Technical level: **quantum Shannon theory** with general $n \geq 1$ and $\varepsilon \geq 0$.

Overview

Main Result: Gaussian Channels I

- Converse bounds for single-mode phase-insensitive bosonic Gaussian channels, most importantly the **photon loss channel**

$$\mathcal{L}_\eta : \ \hat{b} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e} \tag{2}$$

where transmissivity $\eta \in [0, 1]$ and environment in vacuum state.

Main Result: Gaussian Channels I

- Converse bounds for single-mode phase-insensitive bosonic Gaussian channels, most importantly the **photon loss channel**

$$\mathcal{L}_\eta : \ \hat{b} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e} \tag{2}$$

where transmissivity $\eta \in [0,1]$ and environment in vacuum state.

- Previous asymptotic result from [Pirandola *et al.* 2015] in the infinite energy limit

$$P^{\leftrightarrow}(\mathcal{L}_\eta) := \lim_{\varepsilon\to 0}\lim_{n\to\infty} \hat{P}^{\leftrightarrow}_{\mathcal{N}_\eta}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right), \tag{3}$$

which is actually tight in the asymptotic limit, i.e., $P^{\leftrightarrow}(\mathcal{N}_\eta) = \log\left(\frac{1}{1-\eta}\right)$.

Main Result: Gaussian Channels I

- Converse bounds for single-mode phase-insensitive bosonic Gaussian channels, most importantly the **photon loss channel**

$$\mathcal{L}_\eta : \ \hat{b} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e} \tag{2}$$

where transmissivity $\eta \in [0,1]$ and environment in vacuum state.

- Previous asymptotic result from [Pirandola *et al.* 2015] in the infinite energy limit

$$P^{\leftrightarrow}(\mathcal{L}_\eta) := \lim_{\varepsilon\to 0}\lim_{n\to\infty} \hat{P}^{\leftrightarrow}_{\mathcal{N}_\eta}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right), \tag{3}$$

which is actually tight in the asymptotic limit, i.e., $P^{\leftrightarrow}(\mathcal{N}_\eta) = \log\left(\frac{1}{1-\eta}\right)$.

- Drawback: an asymptotic statement, and thus says **little for practical protocols** (called a weak converse bound).

Main Result: Gaussian Channels II

- We show the **non-asymptotic converse bound**

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n}\,, \tag{4}$$

where $C(\varepsilon) := \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ (other choices possible).

Main Result: Gaussian Channels II

- We show the **non-asymptotic converse bound**

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n}\,, \tag{4}$$

where $C(\varepsilon) := \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ (other choices possible).

- Can be used to **assess the performance of any practical quantum repeater** which uses a loss channel n times for desired security ε.

Main Result: Gaussian Channels II

- We show the **non-asymptotic converse bound**

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n}\,, \qquad (4)$$

where $C(\varepsilon) := \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ (other choices possible).

- Can be used to **assess the performance of any practical quantum repeater** which uses a loss channel n times for desired security ε.
- Other variations of this bound are possible if η is not the same for each channel use, if η is chosen adversarially, etc.

Main Result: Gaussian Channels II

- We show the **non-asymptotic converse bound**

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n}\,, \qquad (4)$$

where $C(\varepsilon) := \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ (other choices possible).

- Can be used to **assess the performance of any practical quantum repeater** which uses a loss channel n times for desired security ε.
- Other variations of this bound are possible if η is not the same for each channel use, if η is chosen adversarially, etc.
- We give similar bounds for the quantum-limited amplifier channel (tight), thermalizing channels, amplifier channels, and additive noise channels.

Main Result: Dephasing Channels I

- Previous asymptotic result for the **qubit dephasing channel** $\mathcal{Z}_\gamma : \rho \mapsto (1-\gamma)\,\rho + \gamma Z \rho Z$ with $\gamma \in (0,1)$ is [Bennett *et al.* 1996, Pirandola *et al.* 2015]

$$P^{\leftrightarrow}(\mathcal{Z}_\gamma) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \hat{P}^{\leftrightarrow}_{\mathcal{Z}_\gamma}(n,\varepsilon) = 1 - h(\gamma)\,, \tag{5}$$

with the binary entropy $h(\gamma) := -\gamma \log \gamma - (1-\gamma)\log(1-\gamma)$.

Main Result: Dephasing Channels I

- Previous asymptotic result for the **qubit dephasing channel** $\mathcal{Z}_\gamma : \rho \mapsto (1-\gamma)\,\rho + \gamma Z \rho Z$ with $\gamma \in (0,1)$ is [Bennett *et al.* 1996, Pirandola *et al.* 2015]

$$P^{\leftrightarrow}(\mathcal{Z}_\gamma) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \hat{P}^{\leftrightarrow}_{\mathcal{Z}_\gamma}(n, \varepsilon) = 1 - h(\gamma)\,, \tag{5}$$

with the binary entropy $h(\gamma) := -\gamma \log \gamma - (1-\gamma) \log(1-\gamma)$.

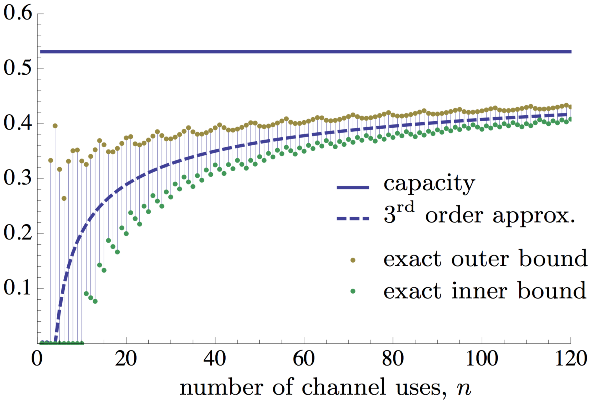
- By combining with [Tomamichel *et al.* 2016] we show the expansion

$$\boxed{\hat{P}^{\leftrightarrow}_{\mathcal{Z}_\gamma}(n, \varepsilon) = 1 - h(\gamma) + \sqrt{\frac{v(\gamma)}{n}}\,\Phi^{-1}(\varepsilon) + \frac{\log n}{2n} + O\left(\frac{1}{n}\right)\,, \qquad (6)}$$

with Φ the cumulative standard Gaussian distribution and the binary entropy variance $v(\gamma) := \gamma(\log \gamma + h(\gamma))^2 + (1-\gamma)(\log(1-\gamma) + h(\gamma))^2$.

Main Result: Dephasing Channels II

- For the dephasing parameter $\gamma = 0.1$ we get (figure from [Tomamichel *et al.* 2016]):



(c) Comparison of strict bounds with third order approximation for $\varepsilon = 5\%$.

Main Result: Erasure Channels

- For the **qubit erasure channel** $\mathcal{E}_p : \rho \mapsto (1-p)\rho + p|e\rangle\langle e|$ with $p \in (0,1)$ we show by combining with [Tomamichel *et al.* 2016] the expansion

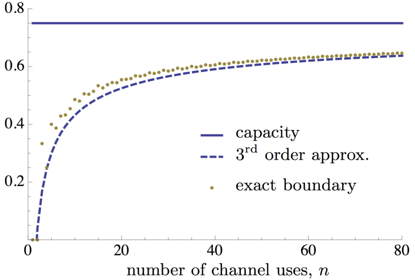
$$\hat{P}^{\leftrightarrow}_{\mathcal{E}_p}(n,\varepsilon) = 1 - p + \sqrt{\frac{p(1-p)}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{1}{n}\right) . \tag{7}$$

Main Result: Erasure Channels

- For the **qubit erasure channel** $\mathcal{E}_p : \rho \mapsto (1-p)\rho + p|e\rangle\langle e|$ with $p \in (0,1)$ we show by combining with [Tomamichel *et al.* 2016] the expansion

$$\hat{P}^{\leftrightarrow}_{\mathcal{E}_p}(n,\varepsilon) = 1 - p + \sqrt{\frac{p(1-p)}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{1}{n}\right) . \tag{7}$$

- For the erasure parameter $p = 0.25$ we get for $\varepsilon = 1\%$ (figure from [Tomamichel *et al.* 2016]):



(b) Comparison of exact bounds with third order approximation.

Proof Idea: Meta Converse I

- **Meta converse approach** from classical channel coding [Polyanskiy *et al.* 2010], uses connection to **hypothesis testing**. In the quantum regime, e.g., for classical communication [Tomamichel & Tan 2015] or quantum communication [Tomamichel *et al.* 2014 & 2016]. We extend this approach to **private communication**.

Proof Idea: Meta Converse I

- **Meta converse approach** from classical channel coding [Polyanskiy *et al.* 2010], uses connection to **hypothesis testing**. In the quantum regime, e.g., for classical communication [Tomamichel & Tan 2015] or quantum communication [Tomamichel *et al.* 2014 & 2016]. We extend this approach to **private communication**.
- Hypothesis testing relative entropy defined for a state ρ, positive semi-definite operator σ, and $\varepsilon \in [0,1]$ as

$$D_H^{\varepsilon}(\rho\|\sigma) := -\log \inf \left\{ \mathrm{Tr}[\Lambda\sigma] : 0 \leq \Lambda \leq I \wedge \mathrm{Tr}[\Lambda\rho] \geq 1-\varepsilon \right\}. \quad (8)$$

Proof Idea: Meta Converse I

- **Meta converse approach** from classical channel coding [Polyanskiy *et al.* 2010], uses connection to **hypothesis testing**. In the quantum regime, e.g., for classical communication [Tomamichel & Tan 2015] or quantum communication [Tomamichel *et al.* 2014 & 2016]. We extend this approach to **private communication**.
- Hypothesis testing relative entropy defined for a state ρ, positive semi-definite operator σ, and $\varepsilon \in [0,1]$ as

$$D_H^\varepsilon(\rho\|\sigma) := -\log\inf\left\{\mathrm{Tr}[\Lambda\sigma] : 0 \leq \Lambda \leq I \wedge \mathrm{Tr}[\Lambda\rho] \geq 1-\varepsilon\right\}. \tag{8}$$

- The ε-relative entropy of entanglement is defined as

$$E_R^\varepsilon(A;B)_\rho := \inf_{\sigma_{AB}\in\mathcal{S}(A:B)} D_H^\varepsilon(\rho_{AB}\|\sigma_{AB}), \tag{9}$$

where $\mathcal{S}(A:B)$ is the set of separable states (cf. relative entropy of entanglement). **Channel's ε-relative entropy of entanglement** is then given as

$$\boxed{E_R^\varepsilon(\mathcal{N}) := \sup_{|\psi\rangle_{AA'}\in\mathcal{H}_{AA'}} E_R^\varepsilon(A;B)_\rho, \tag{10}}$$

where $\rho_{AB} := \mathcal{N}_{A'\to B}(\psi_{AA'})$.

Proof Idea: Meta Converse II

- Goal is the creation of $\log K$ **bits of key**, i.e., states γ_{ABE} with

$$(\mathcal{M}_A \otimes \mathcal{M}_B)(\gamma_{ABE}) = \frac{1}{K} \sum_i |i\rangle\langle i|_A \otimes |i\rangle\langle i|_B \otimes \sigma_E \qquad (11)$$

for some state σ_E and measurement channels $\mathcal{M}_A, \mathcal{M}_B$.

Proof Idea: Meta Converse II

- Goal is the creation of $\log K$ **bits of key**, i.e., states γ_{ABE} with

$$(\mathcal{M}_A \otimes \mathcal{M}_B)(\gamma_{ABE}) = \frac{1}{K} \sum_i |i\rangle\langle i|_A \otimes |i\rangle\langle i|_B \otimes \sigma_E \qquad (11)$$

  for some state σ_E and measurement channels $\mathcal{M}_A, \mathcal{M}_B$.

- In **one-to-one correspondence** with pure states $\gamma_{AA'BB'E}$ such that [Horodecki *et al.* 2005 & 2009]

$$\gamma_{ABA'B'} = U_{ABA'B'}(\Phi_{AB} \otimes \theta_{A'B'})U^{\dagger}_{ABA'B'}\,, \qquad (12)$$

  where Φ_{AB} maximally entangled, $U_{ABA'B'} = \sum_{i,j} |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B \otimes U^{ij}_{A'B'}$ with each $U^{ij}_{A'B'}$ a unitary, and $\theta_{A'B'}$ a state.

Proof Idea: Meta Converse II

- Goal is the creation of $\log K$ **bits of key**, i.e., states γ_{ABE} with

$$(\mathcal{M}_A \otimes \mathcal{M}_B)(\gamma_{ABE}) = \frac{1}{K}\sum_i |i\rangle\langle i|_A \otimes |i\rangle\langle i|_B \otimes \sigma_E \qquad (11)$$

for some state σ_E and measurement channels $\mathcal{M}_A, \mathcal{M}_B$.

- In **one-to-one correspondence** with pure states $\gamma_{AA'BB'E}$ such that [Horodecki *et al.* 2005 & 2009]

$$\gamma_{ABA'B'} = U_{ABA'B'}(\Phi_{AB} \otimes \theta_{A'B'})U^{\dagger}_{ABA'B'}\,, \qquad (12)$$

where Φ_{AB} maximally entangled, $U_{ABA'B'} = \sum_{i,j} |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B \otimes U^{ij}_{A'B'}$ with each $U^{ij}_{A'B'}$ a unitary, and $\theta_{A'B'}$ a state.

- Work in the latter, bipartite picture.

Proof Idea: Meta Converse III

- For **separable states** $\sigma_{AA'BB'}$ (useless for private communication) and a state $\gamma_{AA'BB'}$ with $\log K$ bits of key we have [Horodecki *et al.* 2009]

$$\mathrm{Tr}\{\gamma_{AA'BB'}\sigma_{AA'BB'}\} \leq \frac{1}{K}\,. \tag{13}$$

Proof Idea: Meta Converse III

- For **separable states** $\sigma_{AA'BB'}$ (useless for private communication) and a state $\gamma_{AA'BB'}$ with $\log K$ bits of key we have [Horodecki *et al.* 2009]

$$\mathrm{Tr}\{\gamma_{AA'BB'}\sigma_{AA'BB'}\} \leq \frac{1}{K} \,. \tag{13}$$

- The monotonicity of the channel's ε-relative entropy of entanglement $E_R^\varepsilon(\mathcal{N})$ with respect to LOCC together with (13) implies the **meta converse**

$$\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^\varepsilon(\mathcal{N}) \quad \text{(LOCC pre- and post-processing assistance).} \tag{14}$$

For n channel uses this gives

$$\hat{P}_{\mathcal{N}}(n,\varepsilon) \leq \frac{1}{n} E_R^\varepsilon\left(\mathcal{N}^{\otimes n}\right) \,. \tag{15}$$

Proof Idea: Meta Converse III

- For **separable states** $\sigma_{AA'BB'}$ (useless for private communication) and a state $\gamma_{AA'BB'}$ with $\log K$ bits of key we have [Horodecki *et al.* 2009]

$$\mathrm{Tr}\{\gamma_{AA'BB'}\sigma_{AA'BB'}\} \leq \frac{1}{K}\,. \tag{13}$$

- The monotonicity of the channel's ε-relative entropy of entanglement $E_R^\varepsilon(\mathcal{N})$ with respect to LOCC together with (13) implies the **meta converse**

$$\boxed{\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^\varepsilon(\mathcal{N}) \quad \text{(LOCC pre- and post-processing assistance).}} \tag{14}$$

For n channel uses this gives

$$\hat{P}_{\mathcal{N}}(n,\varepsilon) \leq \frac{1}{n}E_R^\varepsilon\left(\mathcal{N}^{\otimes n}\right)\,. \tag{15}$$

- Finite block-length version of **relative entropy of entanglement** upper bound [Horodoecki *et al.* 2005 & 2009].

Proof Idea: Meta Converse III

- For **separable states** $\sigma_{AA'BB'}$ (useless for private communication) and a state $\gamma_{AA'BB'}$ with $\log K$ bits of key we have [Horodecki *et al.* 2009]

$$\mathrm{Tr}\{\gamma_{AA'BB'}\sigma_{AA'BB'}\} \leq \frac{1}{K}\,. \tag{13}$$

- The monotonicity of the channel's ε-relative entropy of entanglement $E_R^{\varepsilon}(\mathcal{N})$ with respect to LOCC together with (13) implies the **meta converse**

$$\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N}) \quad \text{(LOCC pre- and post-processing assistance).} \tag{14}$$

For n channel uses this gives

$$\hat{P}_{\mathcal{N}}(n,\varepsilon) \leq \frac{1}{n} E_R^{\varepsilon}\left(\mathcal{N}^{\otimes n}\right)\,. \tag{15}$$

- Finite block-length version of **relative entropy of entanglement** upper bound [Horodoecki *et al.* 2005 & 2009].
- The next step is to **evaluate** the meta converse for specific channels of interest.

Proof Idea: Meta Converse IV

- For **teleportation-simulable channels $\mathcal{N}_{A'\to B}$ with associated state ω_{AB}** [Bennett *et al.* 1996, Pirandola *et al.* 2015] the meta converse holds for **general LOCC assistance** and expands as

$$\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon) \leq E_R(A;B)_\omega + \sqrt{\frac{V_{E_R}^{\varepsilon}(A;B)_\omega}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right), \qquad (16)$$

where

$$V_{E_R}^{\varepsilon}(A;B)_\rho \equiv \left\{ \begin{array}{ll} \max_{\sigma_{AB}\in\Pi_{\mathcal{S}}} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon < 1/2 \\ \min_{\sigma_{AB}\in\Pi_{\mathcal{S}}} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon \geq 1/2 \end{array} \right\} \qquad (17)$$

with $\Pi_{\mathcal{S}} \subseteq \mathcal{S}(A:B)$ the set of separable states achieving minimum in the relative entropy of entanglement

$$E_R(A;B)_\rho := \inf_{\sigma_{AB}\in\mathcal{S}(A:B)} D(\rho_{AB}\|\sigma_{AB}). \qquad (18)$$

Here, we have the cumulative standard Gaussian distribution Φ, the relative entropy $D(\rho\|\sigma) := \mathrm{Tr}\left[\rho\left(\log\rho - \log\sigma\right)\right]$, and the relative entropy variance $V(\rho\|\sigma) := \mathrm{Tr}\left[\rho\left(\log\rho - \log\sigma - D(\rho\|\sigma)\right)^2\right]$.

Conclusion

- Our meta converse $\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N})$ gives bounds for the private transmission capabilities of quantum channels. These give the **fundamental limitations of QKD** and thus can be used as **benchmarks for quantum repeaters**.

Conclusion

- Our meta converse $\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N})$ gives bounds for the private transmission capabilities of quantum channels. These give the **fundamental limitations of QKD** and thus can be used as **benchmarks for quantum repeaters**.
- Improve our bound for the **photon loss channel**

$$\hat{P}_{\mathcal{L}_\eta}^{\leftrightarrow}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n} \quad \text{with} \quad C(\varepsilon) = \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{19}$$

to $C'(\varepsilon) := \log\left(\frac{1}{1-\varepsilon}\right)$?

Conclusion

- Our meta converse $\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N})$ gives bounds for the private transmission capabilities of quantum channels. These give the **fundamental limitations of QKD** and thus can be used as **benchmarks for quantum repeaters**.
- Improve our bound for the **photon loss channel**

$$\hat{P}_{\mathcal{L}_\eta}^{\leftrightarrow}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n} \quad \text{with} \quad C(\varepsilon) = \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{19}$$

  to $C'(\varepsilon) := \log\left(\frac{1}{1-\varepsilon}\right)$?
- Corresponding matching **achievability**? (Tight analysis of random coding in infinite dimensions needed.)

Conclusion

- Our meta converse $\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N})$ gives bounds for the private transmission capabilities of quantum channels. These give the **fundamental limitations of QKD** and thus can be used as **benchmarks for quantum repeaters**.
- Improve our bound for the **photon loss channel**

$$\hat{P}_{\mathcal{L}_\eta}^{\leftrightarrow}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n} \quad \text{with} \quad C(\varepsilon) = \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{19}$$

  to $C'(\varepsilon) := \log\left(\frac{1}{1-\varepsilon}\right)$?
- Corresponding matching **achievability**? (Tight analysis of random coding in infinite dimensions needed.)
- Tight **finite-energy** bounds for single-mode phase-insensitive bosonic Gaussian channels?

Conclusion

- Our meta converse $\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N})$ gives bounds for the private transmission capabilities of quantum channels. These give the **fundamental limitations of QKD** and thus can be used as **benchmarks for quantum repeaters**.
- Improve our bound for the **photon loss channel**

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n} \quad \text{with} \quad C(\varepsilon) = \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{19}$$

to $C'(\varepsilon) := \log\left(\frac{1}{1-\varepsilon}\right)$?

- Corresponding matching **achievability**? (Tight analysis of random coding in infinite dimensions needed.)
- Tight **finite-energy** bounds for single-mode phase-insensitive bosonic Gaussian channels?
- Understand more channels, for example such with $P^{\leftrightarrow} > 0$ but zero quantum capacity [Horodecki *et al.* 2008]?

Extra: Gaussian Formulas

- For **Gaussian channels** we need formulas for the relative entropy $D(\rho\|\sigma)$ and the relative entropy variance $V(\rho\|\sigma)$.
- From [Chen 2005, Pirandola *et al.* 2015] and [Wilde *et al.* 2016], respectively: writing zero-mean Gaussian states in exponential form as

$$\rho = Z_\rho^{-1/2} \exp\left\{-\frac{1}{2}\hat{x}^T G_\rho \hat{x}\right\} \quad \text{with} \tag{20}$$

$$Z_\rho := \det(V^\rho + i\Omega/2), \quad G_\rho := 2i\Omega \operatorname{arcoth}(2V^\rho i\Omega)\,, \tag{21}$$

and V^ρ the Wigner function covariance matrix for ρ, we have

$$D(\rho\|\sigma) = \frac{1}{2}\left(\log\left(\frac{Z_\sigma}{Z_\rho}\right) - \mathrm{Tr}\left[\Delta V^\rho\right]\right) \tag{22}$$

$$V(\rho\|\sigma) = \frac{1}{2}\mathrm{Tr}\{\Delta V^\rho \Delta V^\rho\} + \frac{1}{8}\mathrm{Tr}\{\Delta\Omega\Delta\Omega\}\,, \tag{23}$$

where $\Delta := G_\rho - G_\sigma$.