Semidefinite programming hierarchies for quantum adversaries

Mario Berta (IQIM Caltech), Omar Fawzi (ENS Lyon), Volkher Scholz (Ghent University)

(arXiv:1506.08810 - Quantum Bilinear Optimisation)

Overview

- Theoretical talk, plus start with non-cryptographic problem
- Classical noisy channel coding versus entanglement-assisted *channel coding* (**quantum assistance**)
- Semidefinite programming (sdp) hierarchies for understanding (bounding) the difference
- Cryptography: randomness extractors versus *quantum-proof randomness extractors* (**quantum adversary**)
- Conclusion / Outlook

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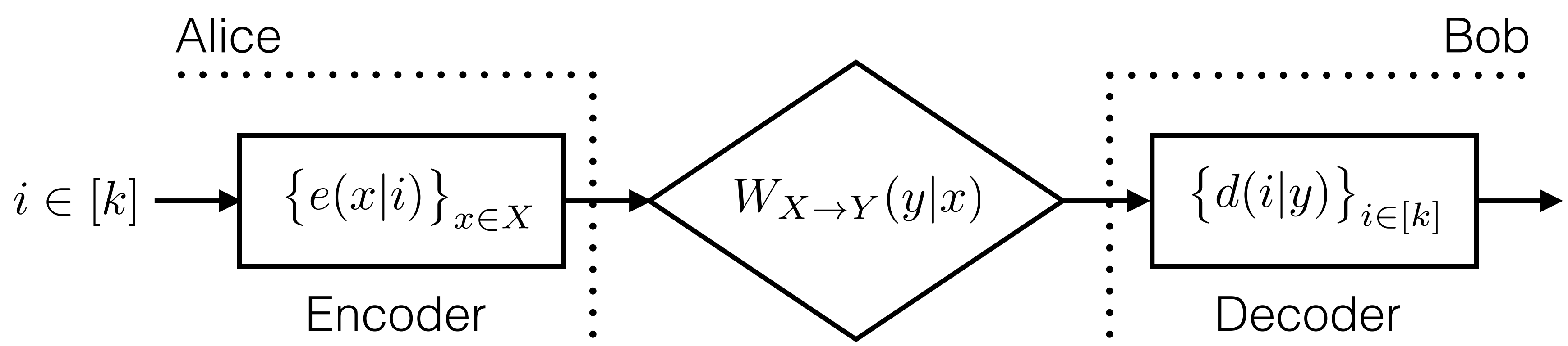
Classical noisy channel coding (I)



- Given noisy channel $W_{X\to Y}$ mapping X to Y with transition probability:

$$W_{X\to Y}(y|x)\ \forall (x,y) \in X \times Y$$

- The goal is to send k different messages using W while minimising the error probability for decoding:

"bilinear optimisation"

$$\begin{aligned} p_{\text{succ}}(W,k) := \underset{(e,d)}{\text{maximize}} \quad & \frac{1}{k} \sum_{x,y,i} W_{X\to Y}(y|x) e(x|i) d(i|y) \\ \text{subject to} \quad & \sum_x e(x|i) = 1 \quad \forall i \in [k]\,, \quad \sum_i d(i|y) = 1 \quad \forall y \in Y \\ & 0 \le e(x|i) \le 1 \quad \forall (x,i) \in X \times [k]\,, \quad 0 \le d(i|y) \le 1 \quad \forall (i,y) \in [k] \times Y\,. \end{aligned}$$

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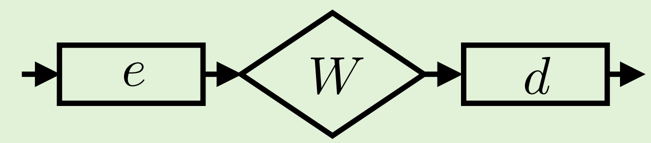

$$\text{subject to} \quad \sum_x e(x|i) = 1 \quad \forall i \in [k]\,, \quad \sum_i d(i|y) = 1 \quad \forall y \in Y$$

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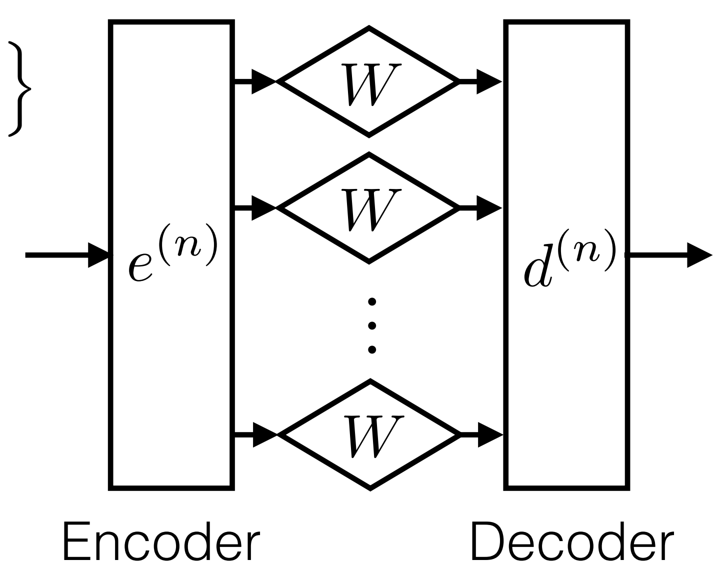
compared to

- Shannon’s asymptotic independent and identical distributed (iid) channel capacity:

Definition: $C(W) := \sup\left\{R \,\middle|\, \forall\delta > 0 : \lim_{n\to\infty} p_{\text{succ}}\left(W^{\times n}, [R(1-\delta)]^n\right) = 1\right\}$

Answer: $C(W) = \max_{P_X} I(X:Y)$ mutual information



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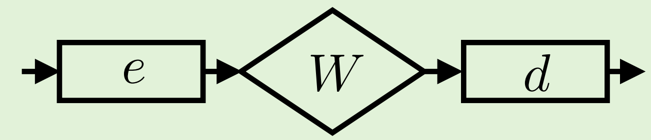

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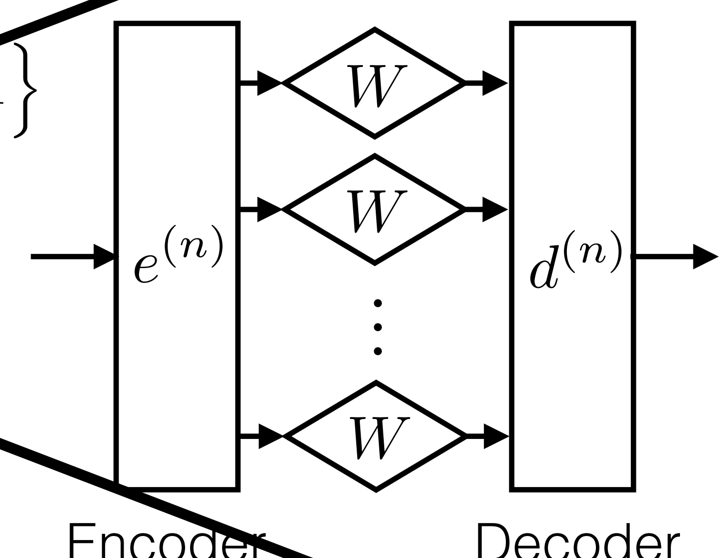
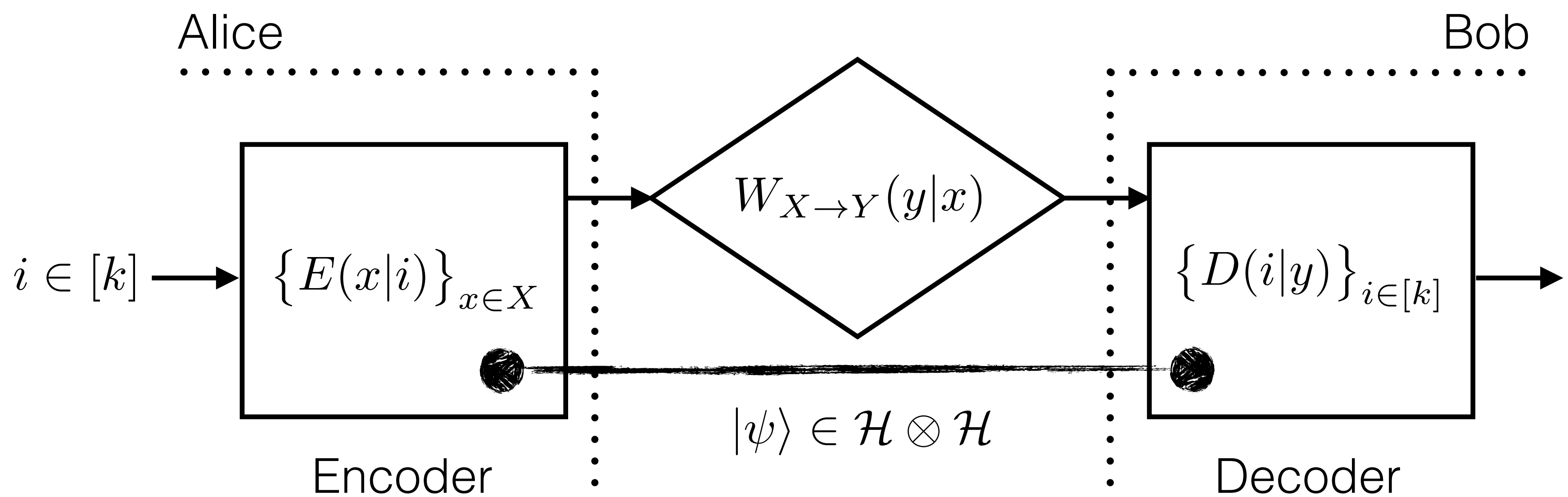


Entanglement-assisted channel coding (I)



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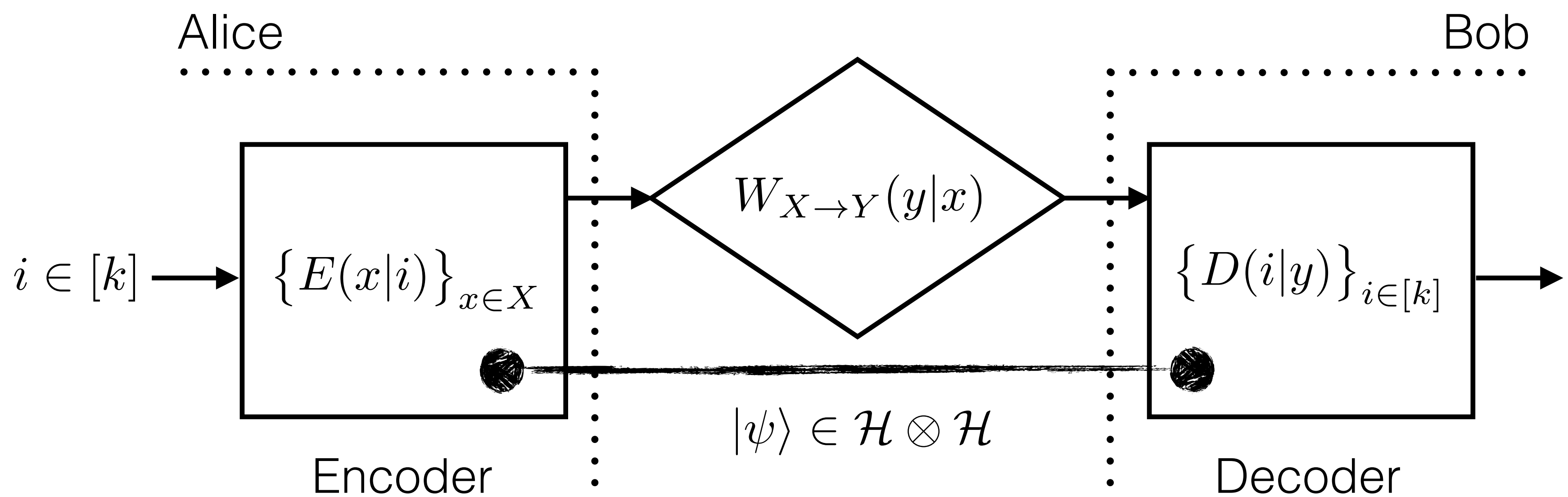

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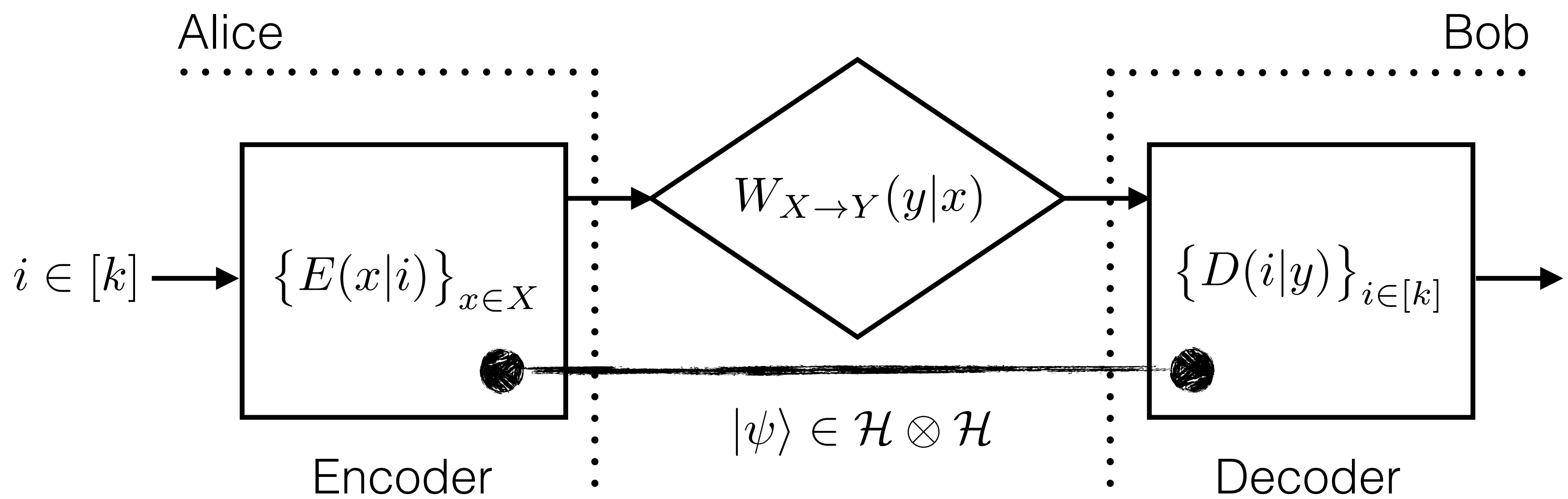

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- Unknown if $p^*_{\mathrm{succ}}(W,k)$ is computable!

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- Understand the possible separation: $p_{\mathrm{succ}}(W,k) \quad \text{versus} \quad p^*_{\mathrm{succ}}(W,k)$

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"linear optimisation with semidefinite constraints"

- We give a **converging hierarchy of semidefinite programming (sdp) relaxations**:

$$p_{\text{succ}}(W,k) \leq p^*_{\text{succ}}(W,k) = \text{sdp}_\infty(W,k) \leq \ldots \leq \text{sdp}_1(W,k)$$ <— *efficiently computable!*

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First level semidefinite programming relaxation (I)

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*motivated by: "**NPA hierarchy**" (Bell inequalities)*

[Lasserre, SIAM (2001)], [Parrilo, Math. Program. (2003)], [Navascues et al., PRL (2007)], [Doherty et al., IEEE CCC (2008)], [Navascues et al., NJP (2008)], [Pironio et al., SIAM (2010)]

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- First step: see as the part of the upper-right block of the Gram matrix

$$\Omega = \sum_{u,v}\langle\psi|X_u X_v|\psi\rangle|u\rangle\langle v| \quad \text{with} \quad X_u = \begin{cases} E(x|i) & u=(i,x) \\ D(j|y) & u=(j,y)\end{cases}$$

for i=j

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- Original constraints can be formulated as positivity conditions on $\Omega:\ \mathrm{sdp}_1(W,k)$

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First level semidefinite programming relaxation (II)

- First level relaxation: $p_{\mathrm{succ}}(W,k) \leq p^*_{\mathrm{succ}}(W,k) \leq \mathrm{sdp}_1(W,k)$

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\mathrm{sdp}_1(W,k) = \underset{\Omega}{\mathrm{maximize}} \quad & \frac{1}{k}\sum_{x,y,i} W_{X\to Y}(y|x)\Omega_{(i,x),(i,y)} \\
\text{subject to} \quad & \Omega \in \mathrm{Pos}(1+k|X|+k|Y|)\,, \quad \Omega_{\emptyset,\emptyset} = 1 \quad \text{with } \emptyset \text{ the empty symbol} \\
& \Omega_{u,v} \geq 0 \quad \forall u,v \in X\times[k]\cup Y\times[k]\cup\{\emptyset\} \\
& \sum_x \Omega_{w,(i,x)} = \Omega_{w,\emptyset} \quad \forall i\in[k]\,, w\in X\times[k]\cup Y\times[k]\cup\{\emptyset\} \\
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- Going back to our example:

$$p_{\rm succ}(Z,2) = \frac{5}{6} \approx 0.833$$

(known before, with two-dimensional assistance)

$$p^*_{\rm succ}(Z,2) \geq \frac{2+2^{-1/2}}{3} \approx 0.902$$

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$$\begin{aligned} \text{sdp}_1(W,k) = \underset{\Omega}{\text{maximize}} \quad & \frac{1}{k}\sum_{x,y,i} W_{X\to Y}(y|x)\Omega_{(i,x),(i,y)} \\ \text{subject to} \quad & \Omega \in \text{Pos}(1+k|X|+k|Y|)\,, \quad \Omega_{\emptyset,\emptyset}=1 \quad \text{with } \emptyset \text{ the empty symbol} \\ & \Omega_{u,v} \geq 0 \quad \forall u,v \in X\times[k]\cup Y\times[k]\cup\{\emptyset\} \\ & \sum_x \Omega_{w,(i,x)} = \Omega_{w,\emptyset} \quad \forall i\in[k]\,, w\in X\times[k]\cup Y\times[k]\cup\{\emptyset\} \\ & \sum_i \Omega_{w,(i,y)} = \Omega_{w,\emptyset} \quad \forall y\in Y, w\in X\times[k]\cup Y\times[k]\cup\{\emptyset\}\,. \end{aligned}$$

- Going back to our example:

$$p_{\text{succ}}(Z,2) = \frac{5}{6} \approx 0.833$$

$$p^*_{\text{succ}}(Z,2) \geq \frac{2+2^{-1/2}}{3} \approx 0.902$$

(known before, with two-dimensional assistance)

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—> further work [Barman and Fawzi., arXiv:1508.04095]

Overview

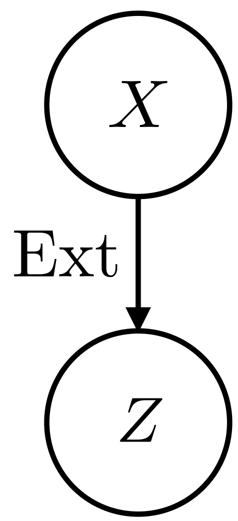
- Theoretical talk, plus start with non-cryptographic problem
- Classical noisy channel coding versus entanglement-assisted *channel coding* (**quantum assistance**)
- Semidefinite programming (sdp) hierarchies for understanding (bounding) the difference
- Cryptography: randomness extractors versus *quantum-proof randomness extractors* (**quantum adversary**)
- Conclusion / Outlook

Quantum Cryptography (I)

- Privacy amplification:

weak source of randomness $X \in \{0,1\}^n$

(with min-entropy k)



uniform random bits $Z \in \{0,1\}^m$

(up to $\epsilon \geq 0$)

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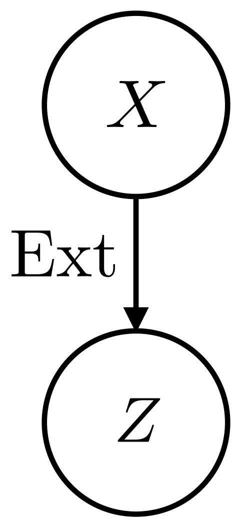

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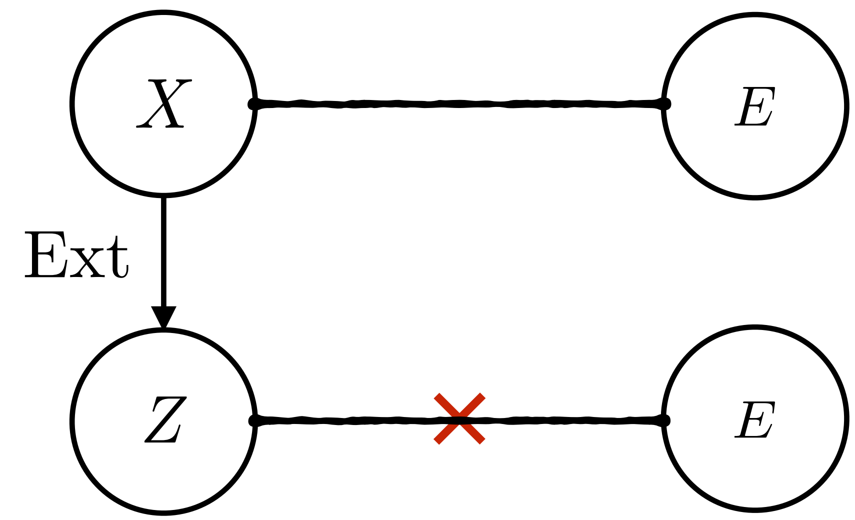
weak source of randomness relative to E



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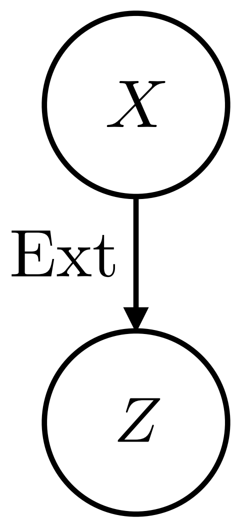

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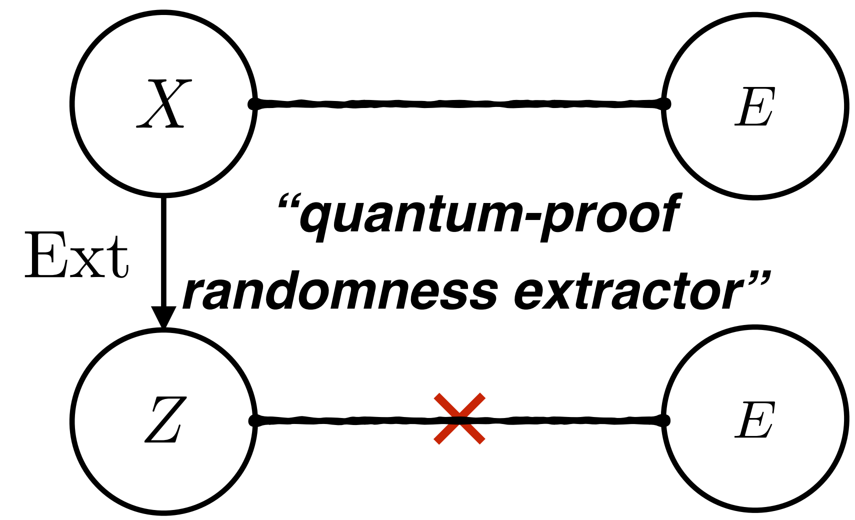

"quantum-proof randomness extractor"

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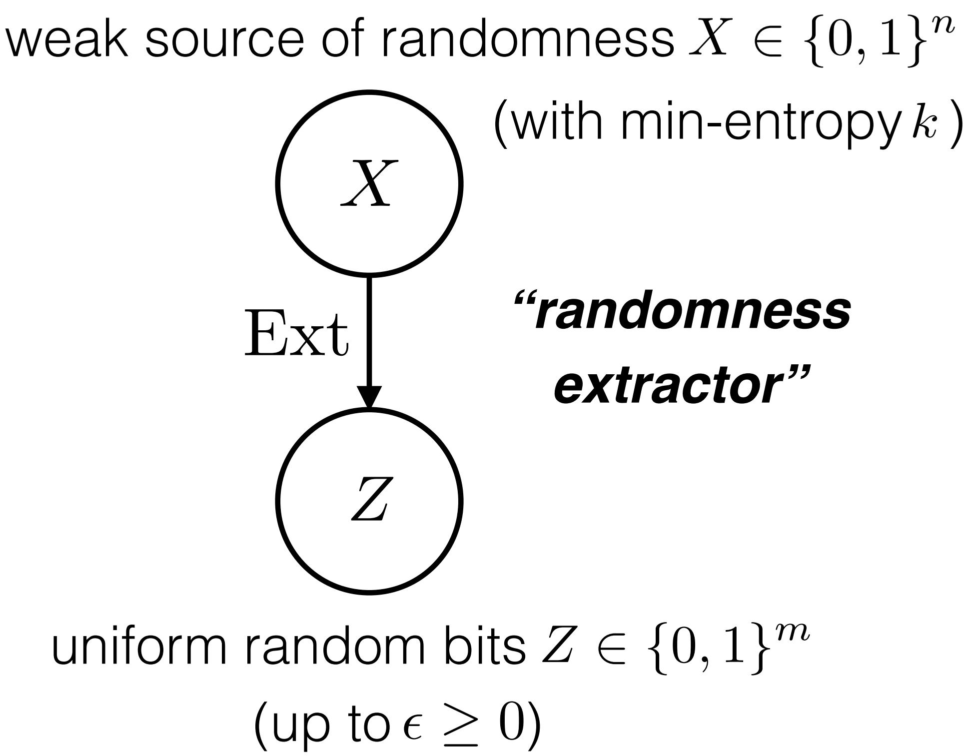

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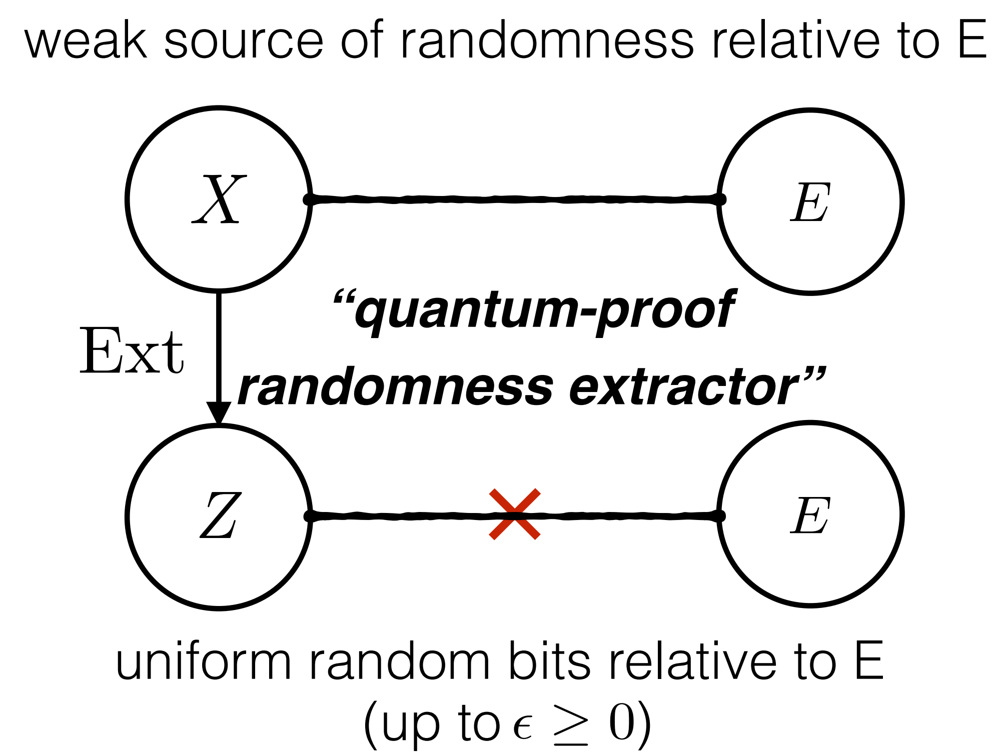

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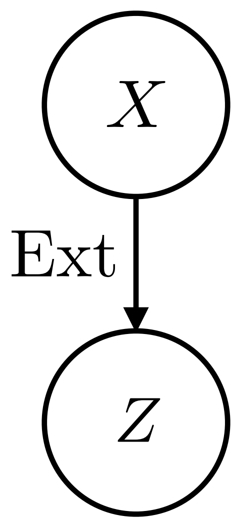

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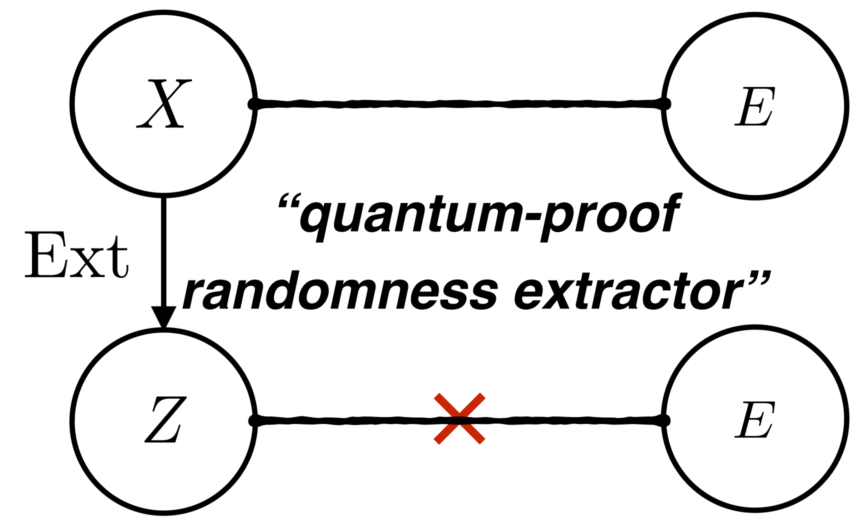

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- Classical versus quantum error (the ϵ): $C(\mathrm{Ext}, k) \quad \text{versus} \quad Q(\mathrm{Ext}, k)$ *computable?*

Quantum Cryptography (II)

- For randomness extractors: classical versus **quantum adversaries**

$C(\mathrm{Ext}, k)$ versus $Q(\mathrm{Ext}, k)$ *computable?*

classical bilinear optimisation	versus	quantum bilinear optimisation
scalar variables	versus	matrix variables

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—> upper bounding the power of quantum adversaries

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- See our paper for results: arXiv:1506.08810 - Quantum Bilinear Optimisation

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- Two-prover games (**Bell inequalities**): we get tighter hierarchy (than previous work)

  —> first level also in independent work [Sikora and Varvitsiotis, arXiv:1506.07297]

- Optimisations over the **completely positive semidefinite cone**: we get the first hierarchy (quantum graph parameters)

  [Laurent and Piovesan, arXiv:1312.6643]

Thanks!