Identifying the Information Gain of a Quantum Measurement

Mario Berta, Joseph M. Renes, Mark M. Wilde - full version
IEEE Transactions on Information Theory, vol. 60, no. 12, pages 7987-8006, 2014

__Question__: How much information (about the input) is gained by performing a given quantum measurement?

<u>Question</u>: How much information (about the input) is gained by performing a given quantum measurement?

Outline

- Setup - making the question precise (information-theoretically)
- Main result - answer
- Proof ideas
- Conclusions

<u>Question</u>: How much information (about the input) is gained by performing a given quantum measurement?

Outline

- Setup - making the question precise (information-theoretically)
- Main result - answer
- Proof ideas
- Conclusions

<u>Question</u>: How much information (about the input) is gained by performing a given quantum measurement?

Outline

- Setup - making the question precise (information-theoretically)
- Main result - answer
- Proof ideas
- Conclusions

<u>Question</u>: How much information (about the input) is gained by performing a given quantum measurement?

Outline

- Setup - making the question precise (information-theoretically)
- Main result - answer
- Proof ideas
- Conclusions

Question: How much information (about the input) is gained by performing a given quantum measurement?

Outline

- Setup - making the question precise (information-theoretically)
- Main result - answer
- Proof ideas
- Conclusions

Setup (a)

- $\boxed{\text{Source}} \xrightarrow{\rho_A} \boxed{\text{Meas. } \mathcal{M}} \longrightarrow \rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\big[\underbrace{(M_A^x)^\dagger \rho_A M_A^x}_{\rho_A^x}\big]}^{p_x}$

Setup (a)

- $\boxed{\text{Source}} \xrightarrow{\rho_A} \boxed{\text{Meas. } \mathcal{M}} \longrightarrow \rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\big[\underbrace{(M_A^x)^\dagger \rho_A M_A^x}_{\rho_A^x}\big]}^{p_x}$

$\mathcal{M} = \{M_A^x\} \quad \sum_x (M_A^x)^\dagger M_A^x = 1_A$

Setup (a)

- $\boxed{\text{Source}} \xrightarrow{\rho_A} \boxed{\text{Meas. } \mathcal{M}} \longrightarrow \quad \rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\big[\underbrace{(M_A^x)^\dagger\, \rho_A M_A^x}_{\rho_A^x}\big]}^{p_x}$

$\mathcal{M} = \{M_A^x\} \quad \sum_x (M_A^x)^\dagger\, M_A^x = 1_A$

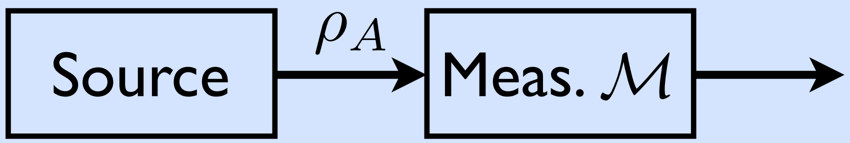
Setup (a)

- Source $\xrightarrow{\rho_A}$ Meas. $\mathcal{M}$ $\longrightarrow$ $\rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\big[\underbrace{(M_A^x)^\dagger \rho_A M_A^x}_{\rho_A^x}\big]}^{p_x}$

  $\mathcal{M} = \{M_A^x\}$ $\sum_x (M_A^x)^\dagger M_A^x = 1_A$

- First guess: amount of data acquired = information gain? $I(\mathcal{M}) \stackrel{?}{=} \log(\#x)$

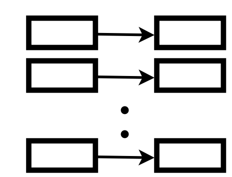
Setup (a)

- 

$\rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\Big[\underbrace{(M_A^x)^\dagger\, \rho_A M_A^x}_{\rho_A^x}\Big]}^{p_x}$

  $\mathcal{M} = \{M_A^x\} \quad \sum_x (M_A^x)^\dagger M_A^x = 1_A$

- First guess: amount of data acquired = information gain? $I(\mathcal{M}) \stackrel{?}{=} \log(\#x)$

  Or for asymptotic independent and identically distributed (iid) setup:  $\mathcal{M}^{\otimes n}(\rho_A^{\otimes n})$

  $$I(\mathcal{M}) \stackrel{?}{=} H(X)_\rho \left(= -\sum_x p_x \log p_x \right)$$

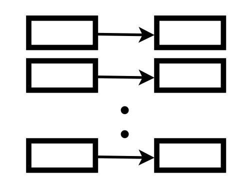
Setup (a)

- $\boxed{\text{Source}} \xrightarrow{\rho_A} \boxed{\text{Meas. } \mathcal{M}} \longrightarrow \rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\Big[\underbrace{(M_A^x)^\dagger \rho_A M_A^x}_{\rho_A^x}\Big]}^{p_x}$

  $\mathcal{M} = \{M_A^x\} \quad \sum_x (M_A^x)^\dagger M_A^x = 1_A$

- First guess: amount of data acquired = information gain? $I(\mathcal{M}) \stackrel{?}{=} \log(\#x)$

  Or for asymptotic independent and identically distributed (iid) setup:

  $\mathcal{M}^{\otimes n}(\rho_A^{\otimes n})$

  $$I(\mathcal{M}) \stackrel{?}{=} H(X)_\rho \left(= -\sum_x p_x \log p_x\right)$$

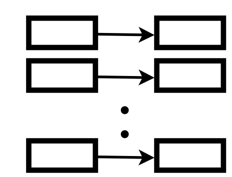
  <u>But</u>: $M_A^x \neq |x\rangle$

Setup (a)

- $$\boxed{\text{Source}} \xrightarrow{\rho_A} \boxed{\text{Meas. } \mathcal{M}} \longrightarrow \rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\big[\underbrace{(M_A^x)^\dagger \rho_A M_A^x}_{\rho_A^x}\big]}^{p_x}$$

$$\mathcal{M} = \{M_A^x\} \quad \sum_x (M_A^x)^\dagger M_A^x = 1_A$$

- First guess: amount of data acquired = information gain? $I(\mathcal{M}) \stackrel{?}{=} \log(\#x)$

  Or for asymptotic independent and identically distributed (iid) setup:  $\mathcal{M}^{\otimes n}(\rho_A^{\otimes n})$

  $$I(\mathcal{M}) \stackrel{?}{=} H(X)_\rho \left(= -\sum_x p_x \log p_x\right)$$

  But: $M_A^x \neq |x\rangle$

- Intuitive reasoning [1]: reduction of entropy? $I(\mathcal{M}) \stackrel{?}{=} H(A)_\rho - \sum_x p_x \cdot H(A)_{\rho^x}$

  $$H(A)_\rho = -\mathrm{tr}[\rho_A \log \rho_A]$$

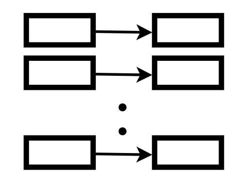
[1] Groenewold, Int. J. of Th. Phys. (1971)

Setup (a)

- Source $\xrightarrow{\rho_A}$ Meas. $\mathcal{M}$ $\longrightarrow$ $\rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\Big[\underbrace{(M_A^x)^\dagger \rho_A M_A^x}_{\rho_A^x}\Big]}^{p_x}$

  $\mathcal{M} = \{M_A^x\}$ $\sum_x (M_A^x)^\dagger M_A^x = 1_A$

- First guess: amount of data acquired = information gain? $I(\mathcal{M}) \stackrel{?}{=} \log(\#x)$

  Or for asymptotic independent and identically distributed (iid) setup:  $\mathcal{M}^{\otimes n}(\rho_A^{\otimes n})$

  $I(\mathcal{M}) \stackrel{?}{=} H(X)_\rho \left(= -\sum_x p_x \log p_x\right)$ <u>But</u>: $M_A^x \neq |x\rangle$

- Intuitive reasoning [1]: reduction of entropy? $I(\mathcal{M}) \stackrel{?}{=} H(A)_\rho - \sum_x p_x \cdot H(A)_{\rho^x}$

- Non-negativity [2]: $H(A)_\rho - \sum_x p_x \cdot H(A)_{\rho^x} \geq 0$ $H(A)_\rho = -\mathrm{tr}[\rho_A \log \rho_A]$

[1] Groenewold, Int. J. of Th. Phys. (1971) [2] Ozawa, J. of Math. Phys. (1986)

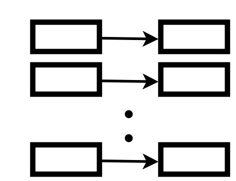
Setup (a)

- Source $\xrightarrow{\rho_A}$ Meas. $\mathcal{M}$ $\longrightarrow$ $\rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\Big[\underbrace{(M_A^x)^\dagger \rho_A M_A^x}_{\rho_A^x}\Big]}^{p_x}$

$\mathcal{M} = \{M_A^x\} \quad \sum_x (M_A^x)^\dagger M_A^x = 1_A$

- First guess: amount of data acquired = information gain? $I(\mathcal{M}) \stackrel{?}{=} \log(\#x)$

  Or for asymptotic independent and identically distributed (iid) setup:

  $\mathcal{M}^{\otimes n}(\rho_A^{\otimes n})$

  $I(\mathcal{M}) \stackrel{?}{=} H(X)_\rho \left(= -\sum_x p_x \log p_x\right)$ **But**: $M_A^x \neq |x\rangle$

- Intuitive reasoning [1]: reduction of entropy? $I(\mathcal{M}) \stackrel{?}{=} H(A)_\rho - \sum_x p_x \cdot H(A)_{\rho^x}$

- Non-negativity [2]: $H(A)_\rho - \sum_x p_x \cdot H(A)_{\rho^x} \geq 0$ $\quad H(A)_\rho = -\mathrm{tr}[\rho_A \log \rho_A]$

- Information and disturbance in quantum measurements [3]:

$$I(\mathcal{M}) \stackrel{?}{=} I(X:R)_\omega \left(= H(X)_\omega + H(R)_\omega - H(XR)_\omega\right) \geq 0$$

$$\omega_{XR} = \sum_x |x\rangle\langle x|_X \otimes \mathrm{tr}_A\Big[(M_A^x \otimes 1_R)^\dagger \underbrace{\rho_{AR}}_{\text{pure}} (M_A^x \otimes 1_R)\Big]$$

[1] Groenewold, Int. J. of Th. Phys. (1971) [2] Ozawa, J. of Math. Phys. (1986) [3] Buscemi et al., Phys. Rev. Lett. (2008)

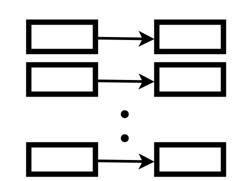
Setup (a)

- $$\boxed{\text{Source}} \xrightarrow{\rho_A} \boxed{\text{Meas. } \mathcal{M}} \longrightarrow \quad \rho_X = \sum_x |x\rangle\langle x|_X \cdot \overbrace{\mathrm{tr}\Big[\underbrace{(M_A^x)^\dagger \rho_A M_A^x}_{\rho_A^x}\Big]}^{p_x}$$

  $$\mathcal{M} = \{M_A^x\} \quad \sum_x (M_A^x)^\dagger M_A^x = 1_A$$

- First guess: amount of data acquired = information gain? $I(\mathcal{M}) \stackrel{?}{=} \log(\#x)$

  Or for asymptotic independent and identically distributed (iid) setup:

  $\mathcal{M}^{\otimes n}(\rho_A^{\otimes n})$

  $$I(\mathcal{M}) \stackrel{?}{=} H(X)_\rho \left(= -\sum_x p_x \log p_x \right)$$

  <u>But</u>: $M_A^x \neq |x\rangle$

- Intuitive reasoning [1]: reduction of entropy? $I(\mathcal{M}) \stackrel{?}{=} H(A)_\rho - \sum_x p_x \cdot H(A)_{\rho^x}$

- Non-negativity [2]: $H(A)_\rho - \sum_x p_x \cdot H(A)_{\rho^x} \geq 0$ $\qquad H(A)_\rho = -\mathrm{tr}[\rho_A \log \rho_A]$

- Information and disturbance in quantum measurements [3]:

  $$I(\mathcal{M}) \stackrel{?}{=} I(X:R)_\omega \left(= H(X)_\omega + H(R)_\omega - H(XR)_\omega\right) \geq 0$$

  $$\omega_{XR} = \sum_x |x\rangle\langle x|_X \otimes \mathrm{tr}_A\Big[(M_A^x \otimes 1_R)^\dagger \underbrace{\rho_{AR}}_{\text{pure}} (M_A^x \otimes 1_R)\Big]$$

- Precise information-theoretic meaning in [4]!

[1] Groenewold, Int. J. of Th. Phys. (1971) [2] Ozawa, J. of Math. Phys. (1986) [3] Buscemi et al., Phys. Rev. Lett. (2008)
[4] Winter, Comm. in Math. Phys. (2004)

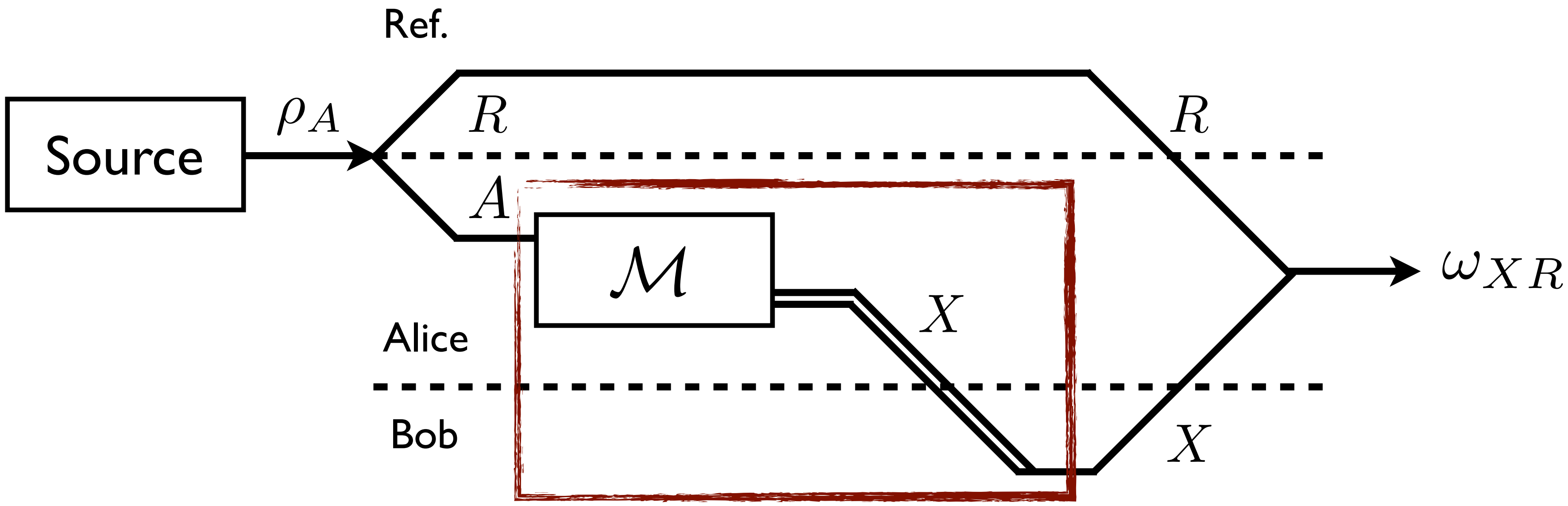
Setup (b)

- Only output data that is correlated to the input should be counted as information gain! Information-theoretic idea of **simulating** the measurement [4]:

[4] Winter, Comm. in Math. Phys. (2004)

Setup (b)

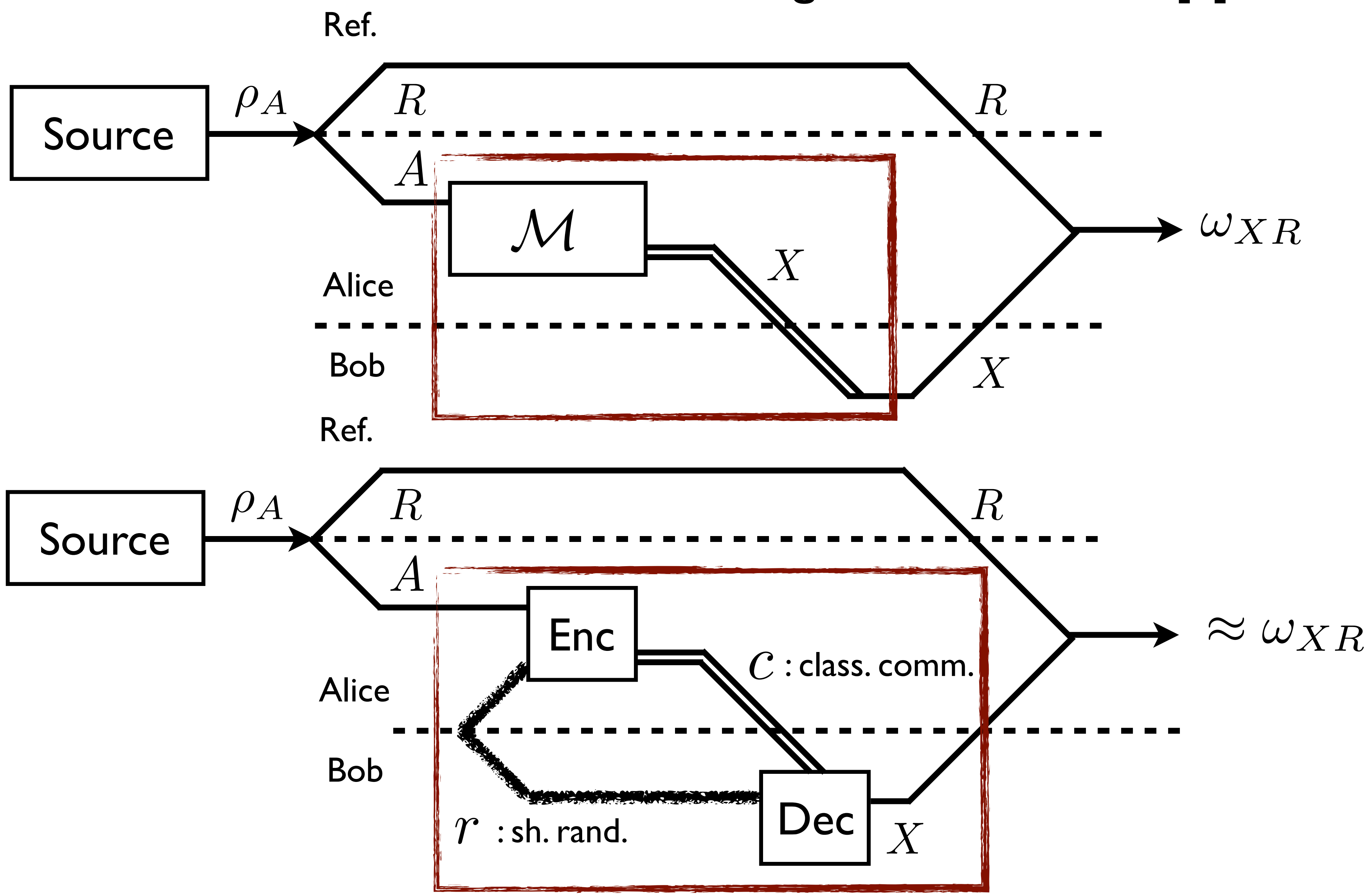
- Only output data that is correlated to the input should be counted as information gain! Information-theoretic idea of **simulating** the measurement [4]:



[4] Winter, Comm. in Math. Phys. (2004)

Setup (b)

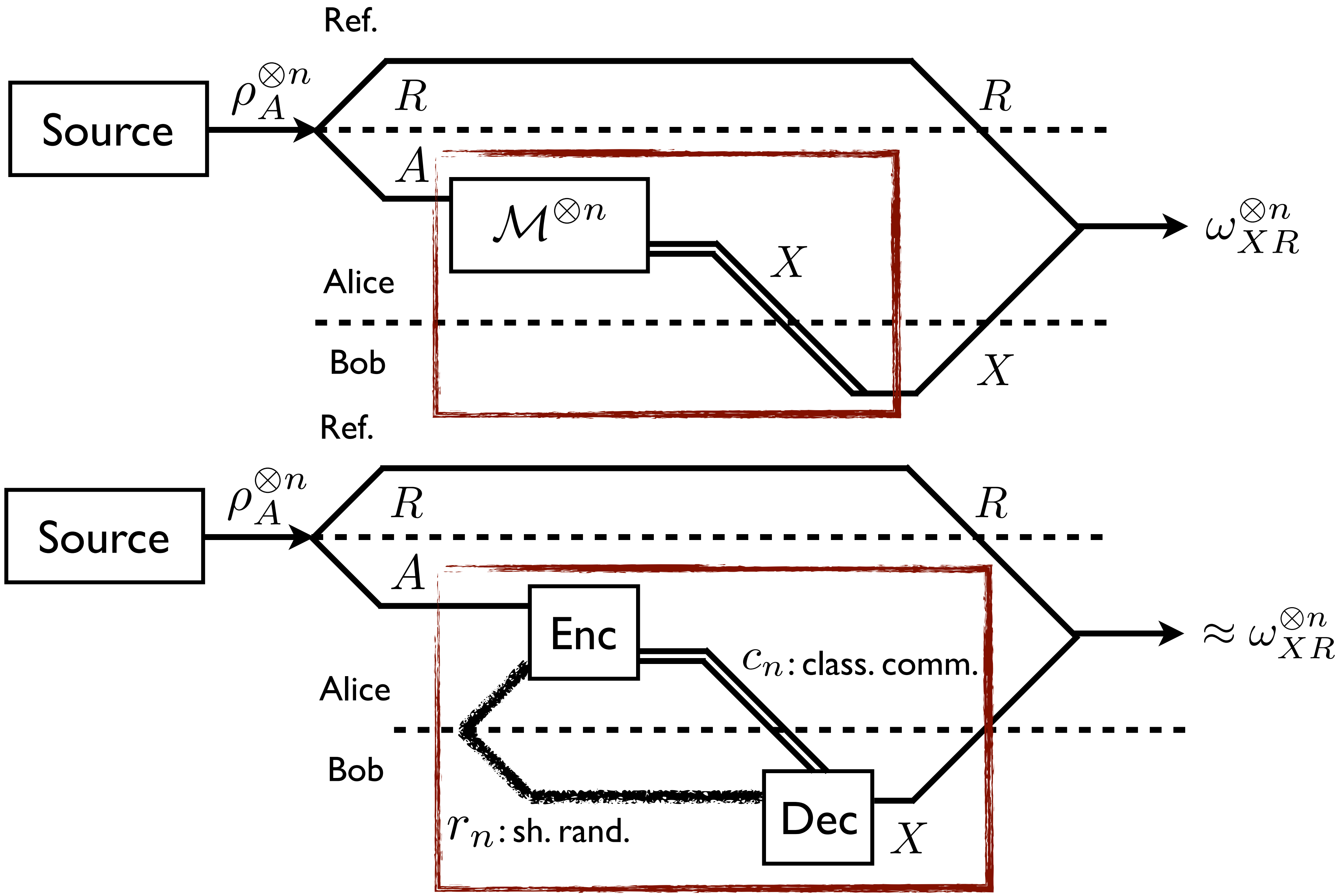
- Only output data that is correlated to the input should be counted as information gain! Information-theoretic idea of **simulating** the measurement [4]:



[4] Winter, Comm. in Math. Phys. (2004)

Setup (b)

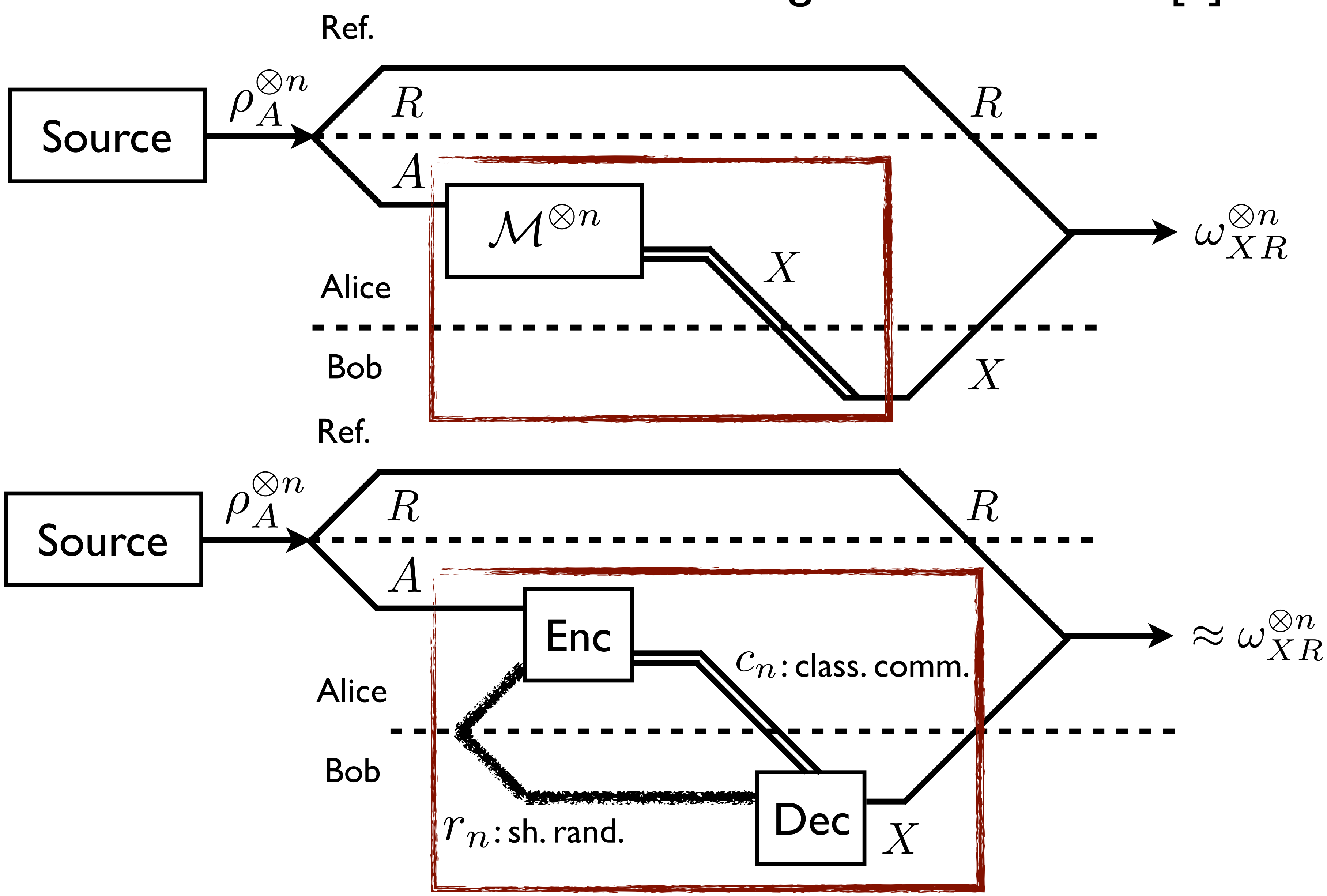
- Only output data that is correlated to the input should be counted as information gain! Information-theoretic idea of **simulating** the measurement [4]:



[4] Winter, Comm. in Math. Phys. (2004)

Setup (b)

- Only output data that is correlated to the input should be counted as information gain! Information-theoretic idea of **simulating** the measurement [4]:



- Main result in [4]:

$$c := \lim_{n\to\infty} \frac{c_n}{n} = I(X:R)_\omega$$

$$r := \lim_{n\to\infty} \frac{r_n}{n} = H(X|R)_\omega \; (= H(XR)_\omega - H(R)_\omega)$$

[4] Winter, Comm. in Math. Phys. (2004)

Setup (c)

- An easy example for [4]:

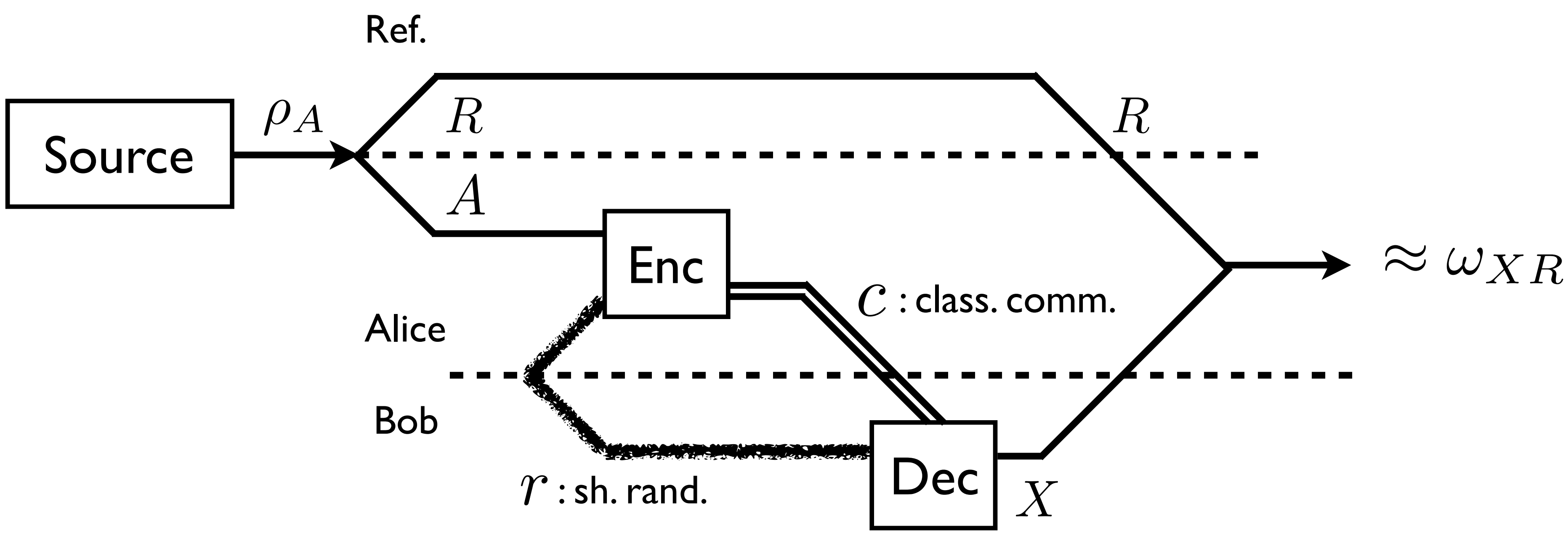
$$\rho_A = \frac{1}{2} 1_A \qquad \mathcal{M} = \left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$

[4] Winter, Comm. in Math. Phys. (2004)

Setup (c)

- An easy example for [4]:

$$\rho_A = \frac{1}{2}1_A \qquad \mathcal{M} = \left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$

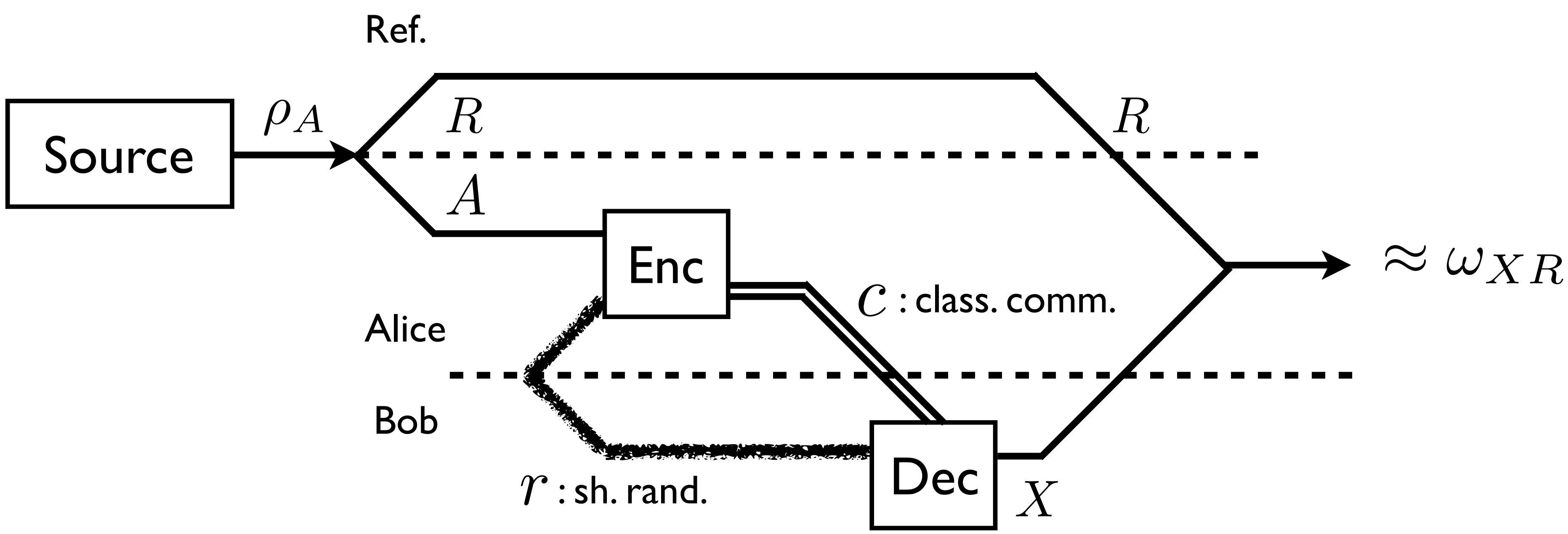


[4] Winter, Comm. in Math. Phys. (2004)

Setup (c)

- An easy example for [4]:

$$\rho_A = \frac{1}{2}1_A \qquad \mathcal{M} = \left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$



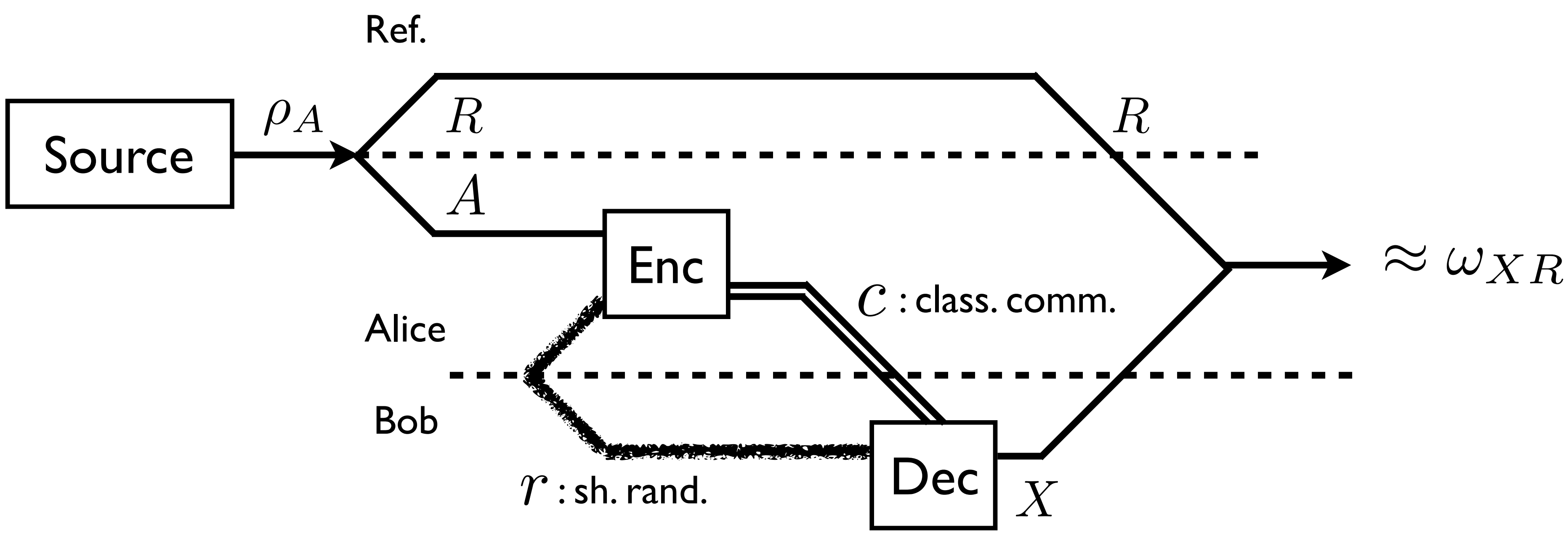
$$\rho_{AR} = |\rho\rangle\langle\rho|_{AR} \quad |\rho\rangle_{AR} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_R + |1\rangle_A \otimes |1\rangle_R\right) = \frac{1}{\sqrt{2}}\left(|+\rangle_A \otimes |+\rangle_R + |-\rangle_A \otimes |-\rangle_R\right)$$

[4] Winter, Comm. in Math. Phys. (2004)

Setup (c)

- An easy example for [4]:

$$\rho_A = \frac{1}{2} 1_A \qquad \mathcal{M} = \left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$



$$\rho_{AR} = |\rho\rangle\langle\rho|_{AR} \quad |\rho\rangle_{AR} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_R + |1\rangle_A \otimes |1\rangle_R\right) = \frac{1}{\sqrt{2}}\left(|+\rangle_A \otimes |+\rangle_R + |-\rangle_A \otimes |-\rangle_R\right)$$

$$\omega_{XR} = \frac{1}{4}\left(|0\rangle\langle 0|_X \otimes |0\rangle\langle 0|_R + |1\rangle\langle 1|_X \otimes |1\rangle\langle 1|_R + |2\rangle\langle 2|_X \otimes |+\rangle\langle +|_R + |3\rangle\langle 3|_X \otimes |-\rangle\langle -|_R\right)$$

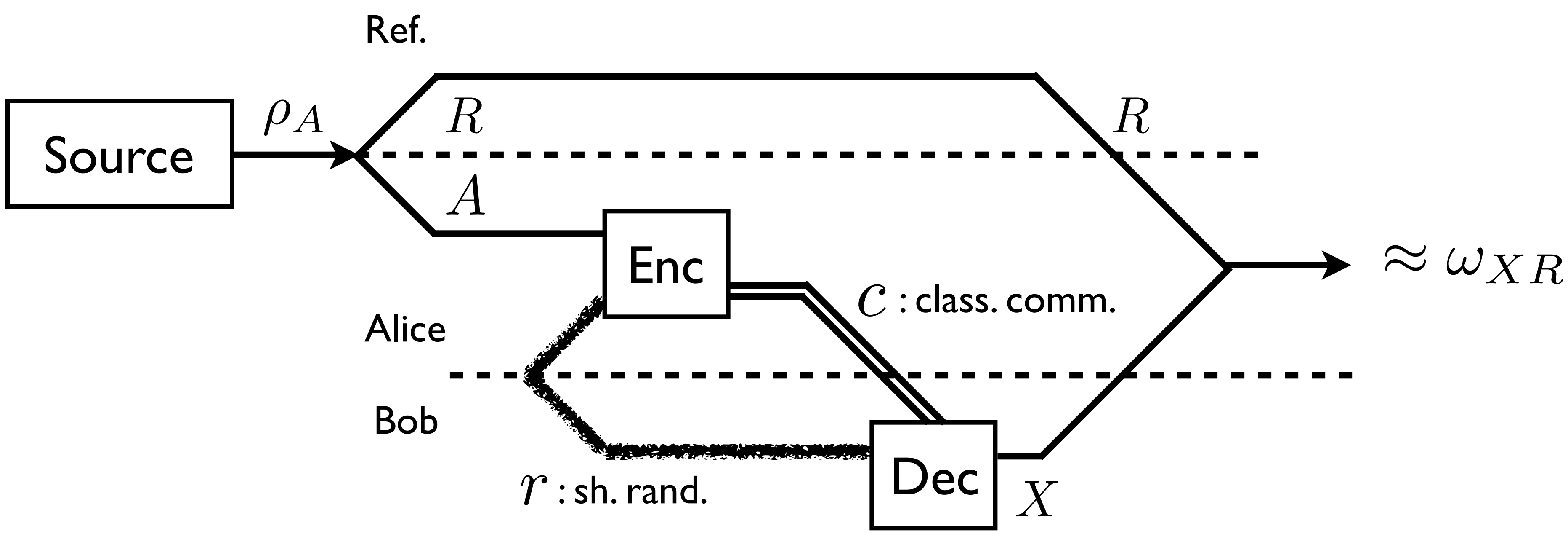
$$\boxed{H(X)_\rho = 2 \quad I(X:R)_\omega = 1 \quad H(X|R)_\omega = 1}$$

[4] Winter, Comm. in Math. Phys. (2004)

Setup (c)

- An easy example for [4]:

$$\rho_A = \frac{1}{2}1_A \qquad \mathcal{M} = \left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$



$$\rho_{AR} = |\rho\rangle\langle\rho|_{AR} \quad |\rho\rangle_{AR} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_R + |1\rangle_A \otimes |1\rangle_R\right) = \frac{1}{\sqrt{2}}\left(|+\rangle_A \otimes |+\rangle_R + |-\rangle_A \otimes |-\rangle_R\right)$$

$$\omega_{XR} = \frac{1}{4}\left(|0\rangle\langle 0|_X \otimes |0\rangle\langle 0|_R + |1\rangle\langle 1|_X \otimes |1\rangle\langle 1|_R + |2\rangle\langle 2|_X \otimes |+\rangle\langle +|_R + |3\rangle\langle 3|_X \otimes |-\rangle\langle -|_R\right)$$

$$\boxed{H(X)_\rho = 2 \quad I(X:R)_\omega = 1 \quad H(X|R)_\omega = 1}$$

- Apply according to shared randomness: $\sigma_X = \left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1| \right\}$ $\sigma_Z = \left\{ \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$

[4] Winter, Comm. in Math. Phys. (2004)

***Question*: How much information (about the input) is gained by performing a given quantum measurement?**

Outline

- Setup - making the question precise (information-theoretically)
- Main result - answer
- Proof ideas
- Conclusions

Main result

- Weaknesses of [4]: (i) only asymptotic iid (ii) **only for known fixed input** (protocol depends on input).

[4] Winter, Comm. in Math. Phys. (2004)

Main result

- Weaknesses of [4]: (i) only asymptotic iid (ii) **only for known fixed input** (protocol depends on input).

- We generalize (ii) to **arbitrarily varying sources** and even **entangled sources** (measurement simulation on general inputs):

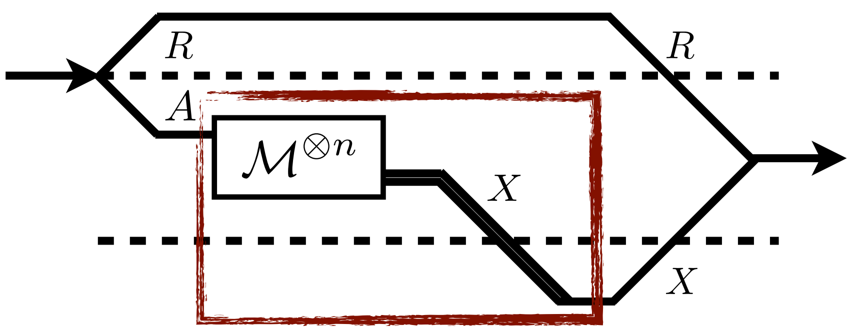
$$\rho_A^{\otimes n} \text{ vs. } \rho_A^1 \otimes \rho_A^2 \otimes \cdots \otimes \rho_A^n \text{ vs. } \rho_{A^n}^n$$

[4] Winter, Comm. in Math. Phys. (2004)

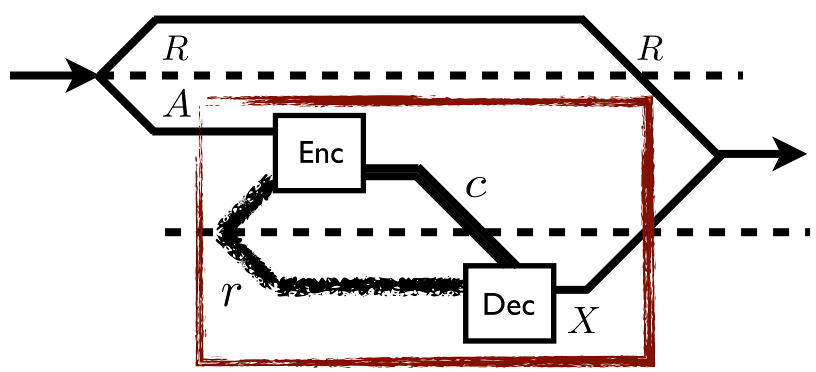
Main result

- Weaknesses of [4]: (i) only asymptotic iid (ii) **only for known fixed input** (protocol depends on input).
- We generalize (ii) to **arbitrarily varying sources** and even **entangled sources** (measurement simulation on general inputs):

$$\rho_A^{\otimes n} \text{ vs. } \rho_A^1 \otimes \rho_A^2 \otimes \cdots \otimes \rho_A^n \text{ vs. } \rho_{A^n}^n$$



$\approx$

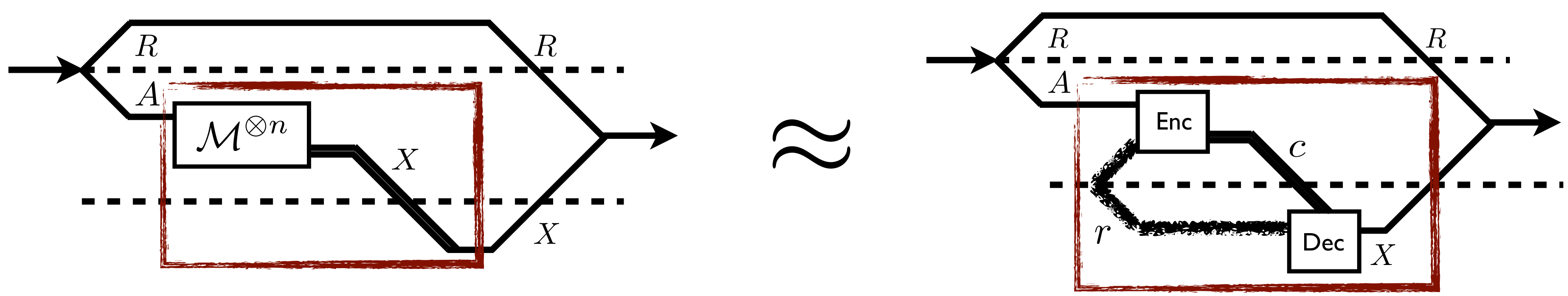


[4] Winter, Comm. in Math. Phys. (2004)

Main result

- Weaknesses of [4]: (i) only asymptotic iid (ii) **only for known fixed input** (protocol depends on input).
- We generalize (ii) to **arbitrarily varying sources** and even **entangled sources** (measurement simulation on general inputs):

$$\rho_A^{\otimes n} \text{ vs. } \rho_A^1 \otimes \rho_A^2 \otimes \cdots \otimes \rho_A^n \text{ vs. } \rho_{A^n}^n$$



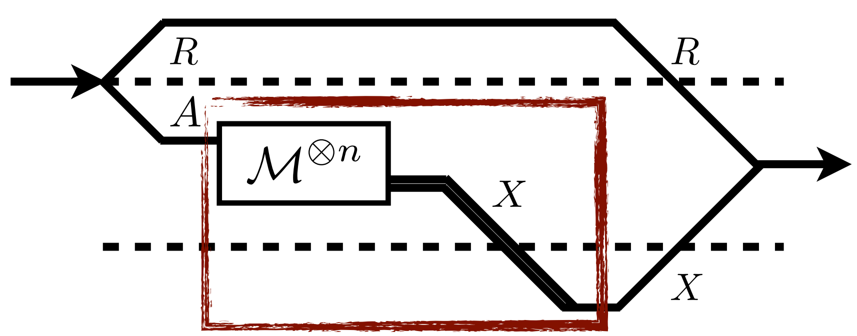
$$c_{\text{avs}} = c_{\text{ent}} = \max_{\rho} I(X:R)_\omega \qquad r = \max_{\rho} H(X)_\omega - \max_{\rho} I(X:R)_\omega$$

[4] Winter, Comm. in Math. Phys. (2004)

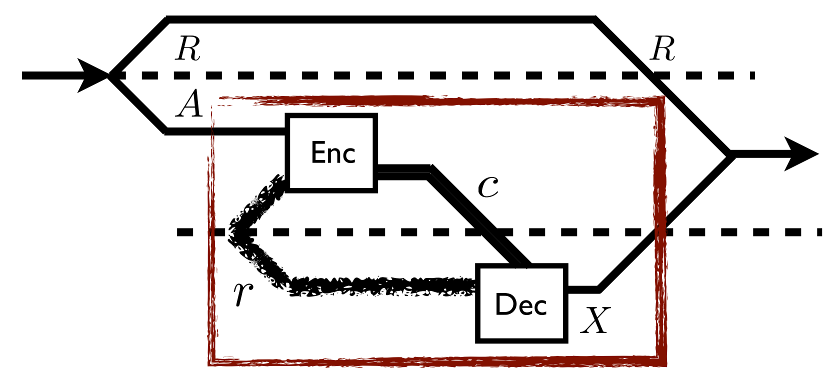
Main result

- Weaknesses of [4]: (i) only asymptotic iid (ii) **only for known fixed input** (protocol depends on input).
- We generalize (ii) to **arbitrarily varying sources** and even **entangled sources** (measurement simulation on general inputs):

$$\rho_A^{\otimes n} \text{ vs. } \rho_A^1 \otimes \rho_A^2 \otimes \cdots \otimes \rho_A^n \text{ vs. } \rho_{A^n}^n$$



$\approx$



$$\boxed{c_{\text{avs}} = c_{\text{ent}} = \max_\rho I(X:R)_\omega \qquad r = \max_\rho H(X)_\omega - \max_\rho I(X:R)_\omega}$$

- Hence, we determined the **information gain of quantum measurements**:

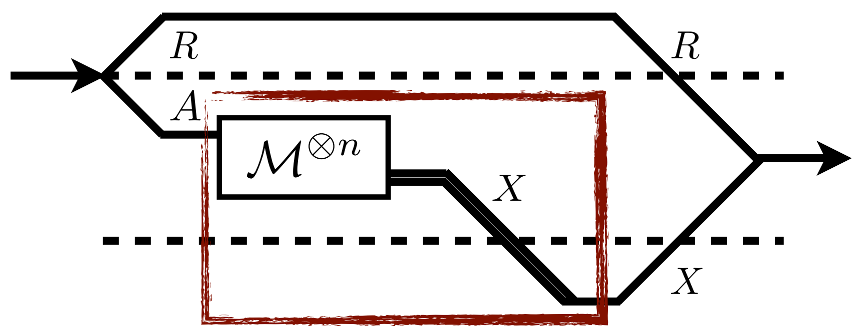
$$\boxed{I(\mathcal{M}) = \max_\rho I(X:R)_\omega}$$

[4] Winter, Comm. in Math. Phys. (2004)

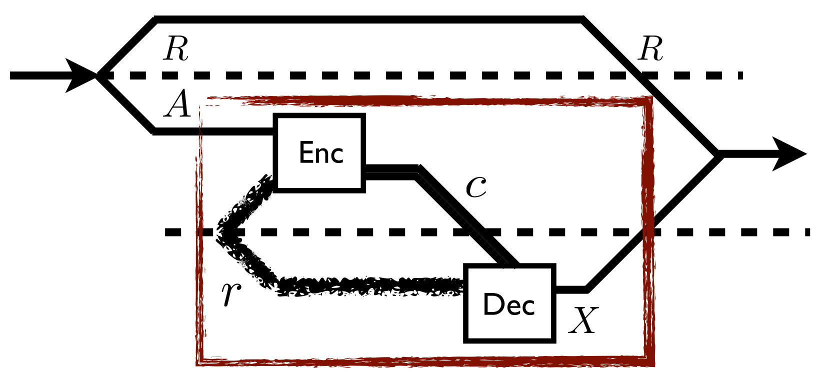
Main result

- Weaknesses of [4]: (i) only asymptotic iid (ii) **only for known fixed input** (protocol depends on input).
- We generalize (ii) to **arbitrarily varying sources** and even **entangled sources** (measurement simulation on general inputs):

$$\rho_A^{\otimes n} \text{ vs. } \rho_A^1 \otimes \rho_A^2 \otimes \cdots \otimes \rho_A^n \text{ vs. } \rho_{A^n}^n$$



$\approx$



$$c_{\text{avs}} = c_{\text{ent}} = \max_\rho I(X:R)_\omega \qquad r = \max_\rho H(X)_\omega - \max_\rho I(X:R)_\omega$$

- Hence, we determined the **information gain of quantum measurements**:

$$I(\mathcal{M}) = \max_\rho I(X:R)_\omega$$

- By proving this, we also generalize (i) to the one-shot case: $c^\epsilon \approx I_{\max}^\epsilon(X:R)_\omega$

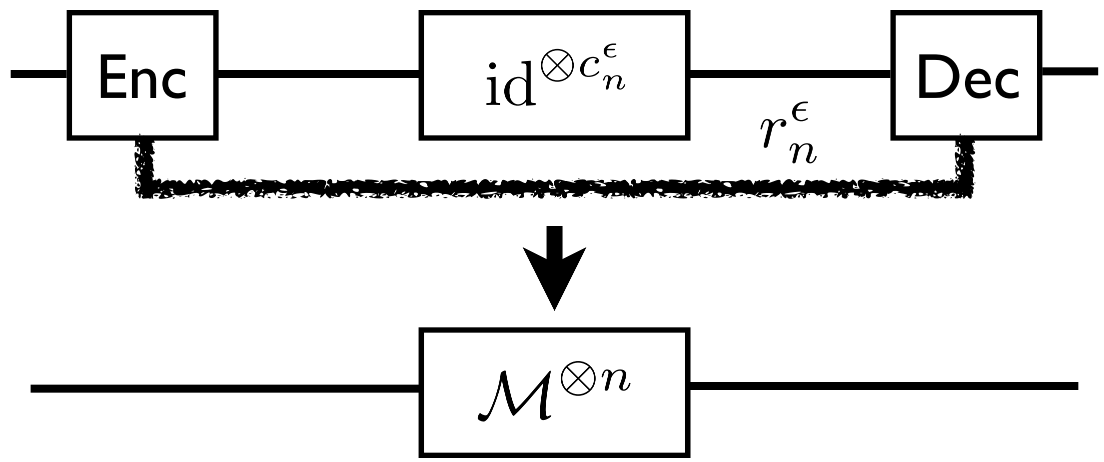
[4] Winter, Comm. in Math. Phys. (2004)

<u>Question</u>*: How much information (about the input) is gained by performing a given quantum measurement?

Outline

- Setup - making the question precise (information-theoretically)
- Main result - answer
- Proof ideas
- Conclusions

Proof Ideas

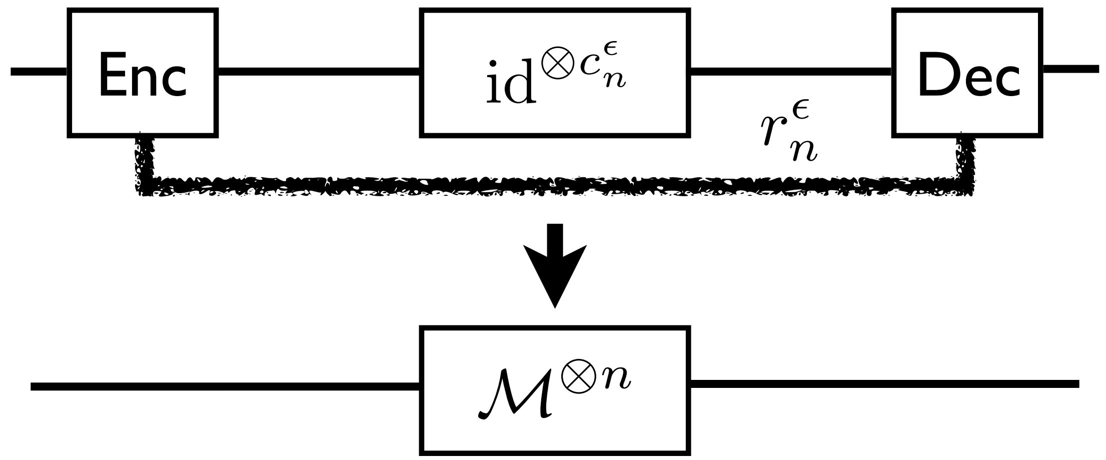
- 

S_n^ϵ : measurement simulation map

c_n^ϵ: classical communication cost

r_n^ϵ: shared randomness cost

Proof Ideas

- 

S_n^ϵ : measurement simulation map

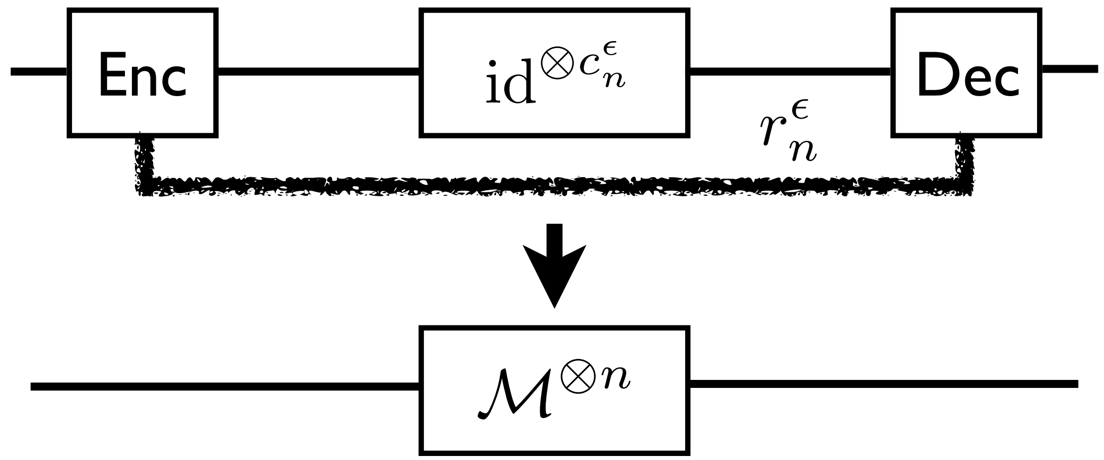
c_n^ϵ : classical communication cost

r_n^ϵ : shared randomness cost

- In [4] for fixed iid input ρ_{AR} : $\left\| \left(\mathcal{M}_A^{\otimes n} \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right) - \left(S_n^\epsilon \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right)\right\|_1 \leq \epsilon$

[4] Winter, Comm. in Math. Phys. (2004)

Proof Ideas

- 

  S_n^ϵ : measurement simulation map

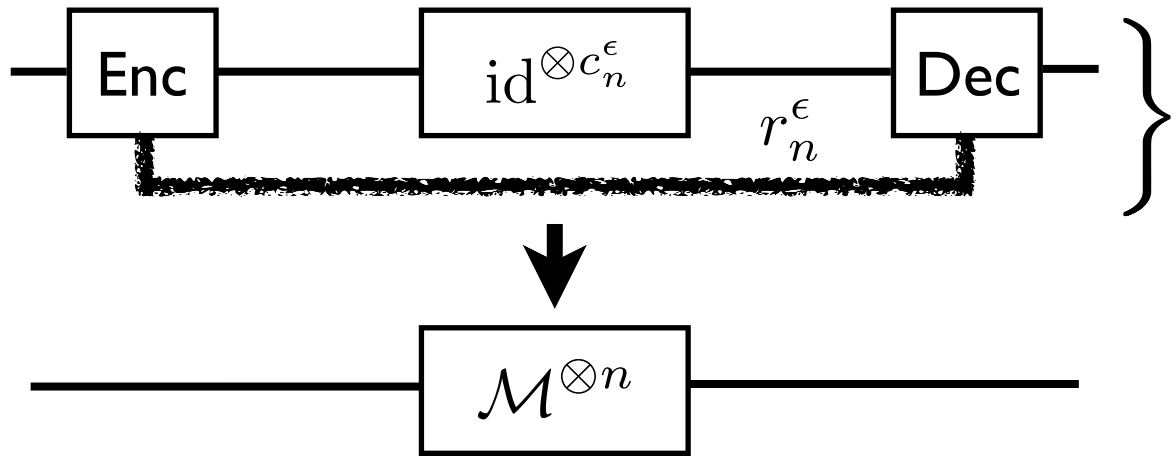
  c_n^ϵ: classical communication cost

  r_n^ϵ: shared randomness cost

- In [4] for fixed iid input ρ_{AR} : $\left\| \left(\mathcal{M}_A^{\otimes n} \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right) - \left(S_n^\epsilon \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right)\right\|_1 \leq \epsilon$
- Now: $\boxed{\left\|\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right\|_\diamond \leq \epsilon}$ $\quad \|\mathcal{M}\|_\diamond = \sup_{k\in\mathbb{N}} \sup_{\|\sigma\|_1\leq 1} \|(\mathcal{M}\otimes \mathrm{id}_k)(\sigma)\|_1$

[4] Winter, Comm. in Math. Phys. (2004)

Proof Ideas

- 

  S_n^ϵ : measurement simulation map

  c_n^ϵ: classical communication cost

  r_n^ϵ: shared randomness cost

- In [4] for fixed iid input ρ_{AR} : $\left\| \left(\mathcal{M}_A^{\otimes n} \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right) - \left(S_n^\epsilon \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right)\right\|_1 \leq \epsilon$

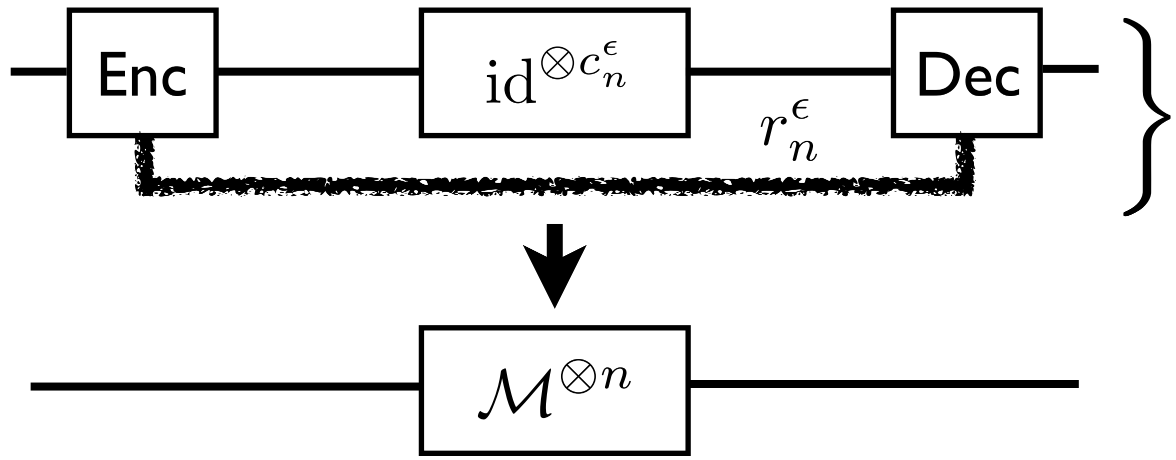
- Now: $\boxed{\left\|\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right\|_\diamond \leq \epsilon}$ $\quad \|\mathcal{M}\|_\diamond = \sup_{k\in\mathbb{N}} \sup_{\|\sigma\|_1 \leq 1} \|(\mathcal{M}\otimes \mathrm{id}_k)(\sigma)\|_1$

- **Post-selection technique** for quantum channels [5]:

$$\boxed{\left\|\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right\|_\diamond \leq \mathrm{poly}(n) \cdot \left\|\left(\left(\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right) \otimes \mathrm{id}_{R'}\right)\left(\zeta_{AR'}^n\right)\right\|_1}$$

[4] Winter, Comm. in Math. Phys. (2004) [5] Christandl et al., Phys. Rev. Lett. (2009)

Proof Ideas

- 

S_n^ϵ : measurement simulation map

c_n^ϵ : classical communication cost

r_n^ϵ : shared randomness cost

- In [4] for fixed iid input ρ_{AR} : $\left\|\left(\mathcal{M}_A^{\otimes n} \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right) - \left(S_n^\epsilon \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right)\right\|_1 \leq \epsilon$

- Now: $\boxed{\left\|\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right\|_\diamond \leq \epsilon}$ $\quad \|\mathcal{M}\|_\diamond = \sup_{k\in\mathbb{N}} \sup_{\|\sigma\|_1 \leq 1} \|(\mathcal{M}\otimes \mathrm{id}_k)(\sigma)\|_1$

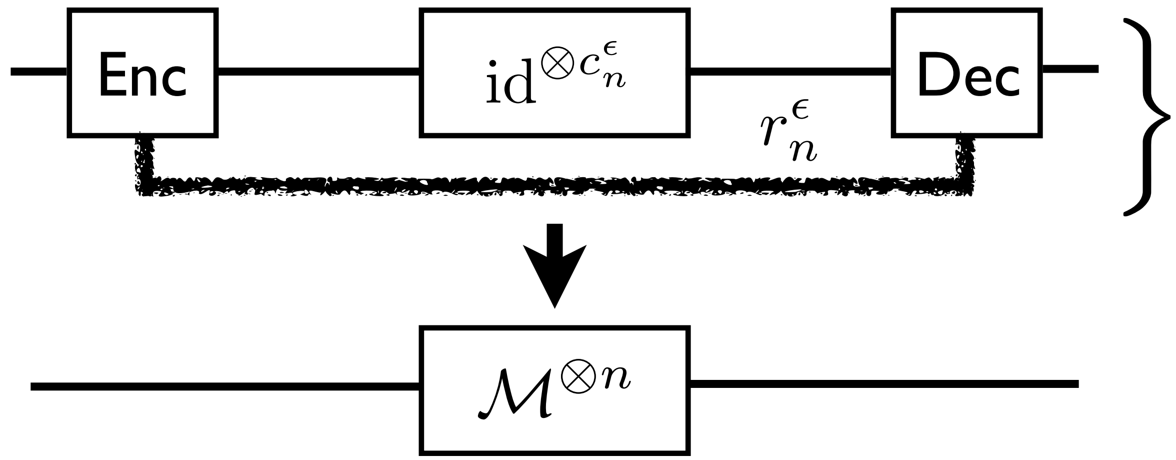
- **Post-selection technique** for quantum channels [5]:

$$\boxed{\left\|\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right\|_\diamond \leq \mathrm{poly}(n)\cdot\left\|\left(\left(\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right)\otimes \mathrm{id}_{R'}\right)\left(\zeta_{AR'}^n\right)\right\|_1}$$

<u>But</u>: $\zeta_{AR'}^n$ purification of a special de Finetti state (a state which consists of n iid copies of a state on a single subsystem), i.e., no iid structure and [4] not applicable!

[4] Winter, Comm. in Math. Phys. (2004) [5] Christandl et al., Phys. Rev. Lett. (2009)

Proof Ideas

- 

S_n^ϵ : measurement simulation map

c_n^ϵ: classical communication cost

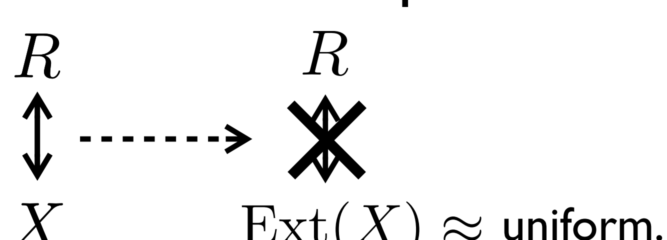
r_n^ϵ: shared randomness cost

- In [4] for fixed iid input ρ_{AR} : $\left\| \left(\mathcal{M}_A^{\otimes n} \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right) - \left(S_n^\epsilon \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right)\right\|_1 \leq \epsilon$
- Now: $\boxed{\left\|\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right\|_\diamond \leq \epsilon}$ $\quad \|\mathcal{M}\|_\diamond = \sup_{k\in\mathbb{N}} \sup_{\|\sigma\|_1\leq 1} \|(\mathcal{M}\otimes \mathrm{id}_k)(\sigma)\|_1$
- **Post-selection technique** for quantum channels [5]:

$$\boxed{\left\|\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right\|_\diamond \leq \mathrm{poly}(n)\cdot\left\|\left(\left(\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right)\otimes \mathrm{id}_{R'}\right)\left(\zeta_{AR'}^n\right)\right\|_1}$$

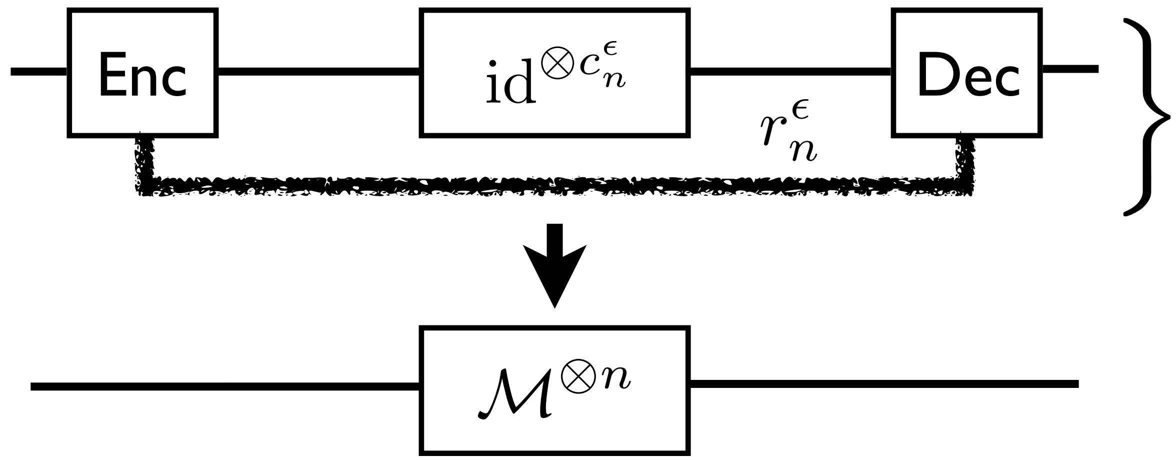
<u>But</u>: $\zeta_{AR'}^n$ purification of a special de Finetti state (a state which consists of n iid copies of a state on a single subsystem), i.e., no iid structure and [4] not applicable!

- New one-shot protocol based on **quantum-proof randomness extractors**:



[4] Winter, Comm. in Math. Phys. (2004) [5] Christandl et al., Phys. Rev. Lett. (2009)

Proof Ideas

- 

  S_n^ϵ : measurement simulation map

  c_n^ϵ : classical communication cost

  r_n^ϵ : shared randomness cost

- In [4] for fixed iid input ρ_{AR} : $\left\|\left(\mathcal{M}_A^{\otimes n} \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right) - \left(S_n^\epsilon \otimes \mathrm{id}_R^{\otimes n}\right)\left(\rho_{AR}^{\otimes n}\right)\right\|_1 \leq \epsilon$

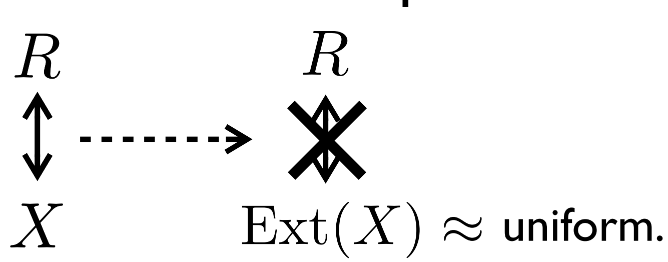
- Now: $\boxed{\left\|\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right\|_\diamond \leq \epsilon}$ $\quad \|\mathcal{M}\|_\diamond = \sup_{k \in \mathbb{N}} \sup_{\|\sigma\|_1 \leq 1} \|(\mathcal{M} \otimes \mathrm{id}_k)(\sigma)\|_1$

- **Post-selection technique** for quantum channels [5]:

$$\boxed{\left\|\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right\|_\diamond \leq \mathrm{poly}(n) \cdot \left\|\left(\left(\mathcal{M}_A^{\otimes n} - S_n^\epsilon\right) \otimes \mathrm{id}_{R'}\right)\left(\zeta_{AR'}^n\right)\right\|_1}$$

  But: $\zeta_{AR'}^n$ purification of a special de Finetti state (a state which consists of n iid copies of a state on a single subsystem), i.e., no iid structure and [4] not applicable!

- New one-shot protocol based on **quantum-proof randomness extractors**:



Idea: extract all randomness from measurement data and only send the rest from Alice to Bob to simulate the measurement

[4] Winter, Comm. in Math. Phys. (2004) [5] Christandl et al., Phys. Rev. Lett. (2009)

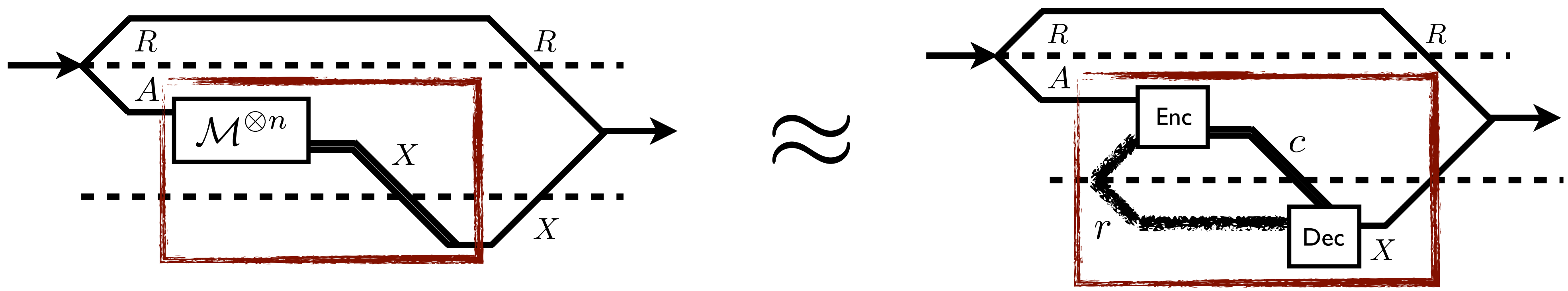
<u>Question</u>: How much information (about the input) is gained by performing a given quantum measurement?

Outline

- Setup - making the question precise (information-theoretically)
- Main result - answer
- Proof ideas
- Conclusions

Conclusions

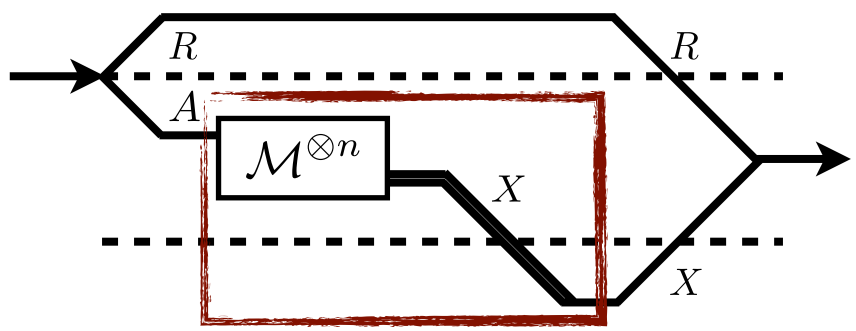
- Our main result is **measurement simulation on general inputs**:



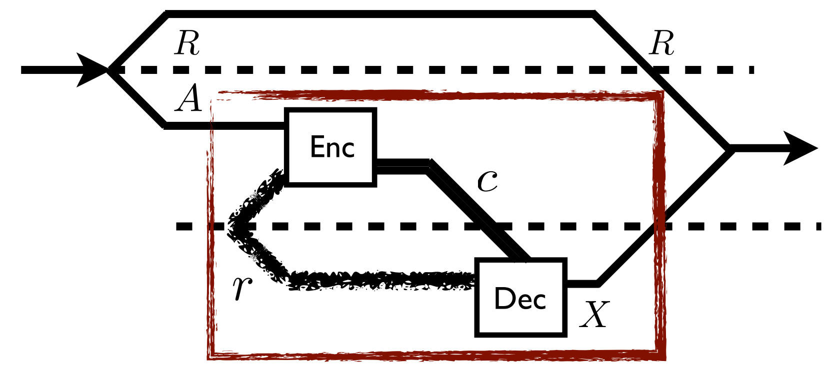
$$c_{\text{avs}} = c_{\text{ent}} = \max_{\rho} I(X:R)_{\omega} \qquad r = \max_{\rho} H(X)_{\omega} - \max_{\rho} I(X:R)_{\omega}$$

Conclusions

- Our main result is **measurement simulation on general inputs**:



$$\approx$$



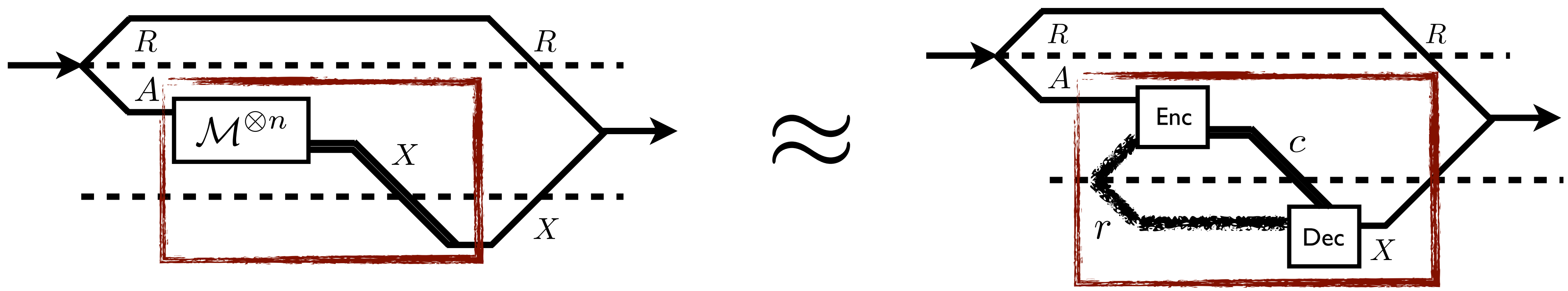
$$\boxed{c_{\text{avs}} = c_{\text{ent}} = \max_{\rho} I(X:R)_{\omega} \qquad r = \max_{\rho} H(X)_{\omega} - \max_{\rho} I(X:R)_{\omega}}$$

- Hence, we determined the **information gain of quantum measurements**:

$$\boxed{I(\mathcal{M}) = \max_{\rho} I(X:R)_{\omega}}$$

Conclusions

- Our main result is **measurement simulation on general inputs**:



$$c_{\mathrm{avs}} = c_{\mathrm{ent}} = \max_{\rho} I(X:R)_{\omega} \qquad r = \max_{\rho} H(X)_{\omega} - \max_{\rho} I(X:R)_{\omega}$$

- Hence, we determined the **information gain of quantum measurements**:

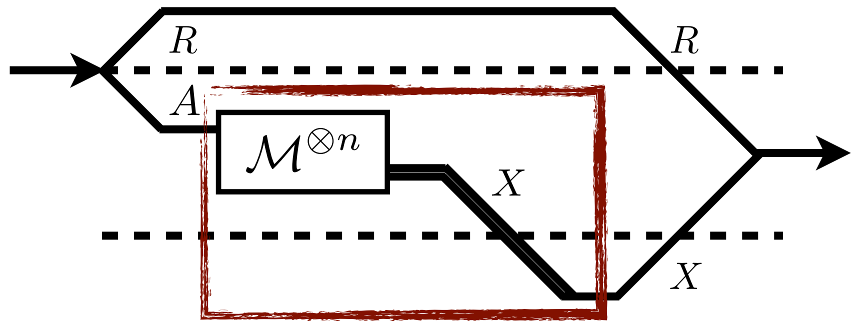
$$I(\mathcal{M}) = \max_{\rho} I(X:R)_{\omega}$$

- Result between classical [6] and quantum [7,8] reverse Shannon theorem

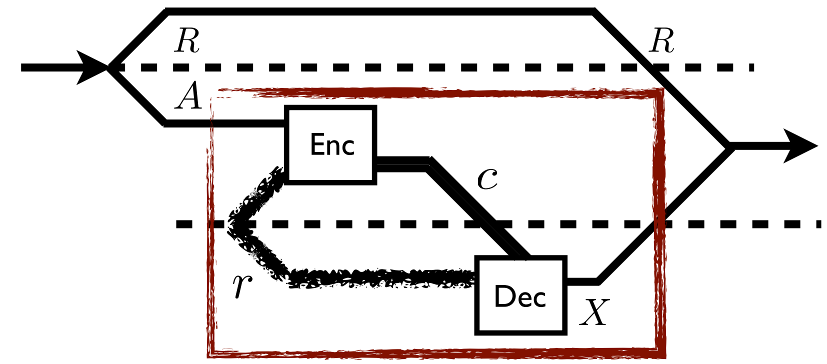
[6] Bennett et al., IEEE Trans. on Inf. Th. (2002) [7] Bennett et al., arXiv:0912.5537v5 [8] Berta et al., Comm. in Math. Phys. (2011)

Conclusions

- Our main result is **measurement simulation on general inputs**:



$\approx$



$$c_{\text{avs}} = c_{\text{ent}} = \max_{\rho} I(X:R)_{\omega} \qquad r = \max_{\rho} H(X)_{\omega} - \max_{\rho} I(X:R)_{\omega}$$

- Hence, we determined the **information gain of quantum measurements**:

$$I(\mathcal{M}) = \max_{\rho} I(X:R)_{\omega}$$

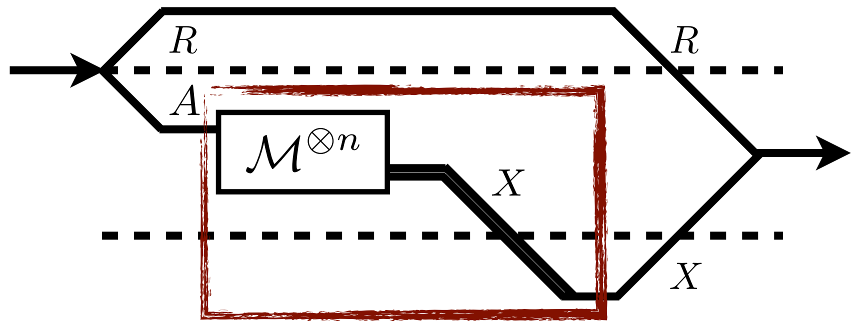
- Result between classical [6] and quantum [7,8] reverse Shannon theorem
- Extension: rate region for non-feedback vs. feedback measurement simulation

$$c(r) = \max\left\{\max_{\rho} I(X:R)_{\omega}, \max_{\rho} H(X)_{\omega} - r\right\}$$

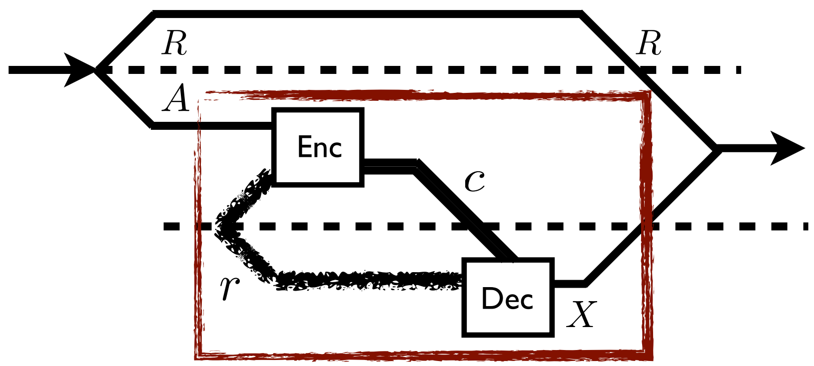
[6] Bennett et al., IEEE Trans. on Inf. Th. (2002) [7] Bennett et al., arXiv:0912.5537v5 [8] Berta et al., Comm. in Math. Phys. (2011)

Conclusions

- Our main result is **measurement simulation on general inputs**:



$\approx$



$$c_{\text{avs}} = c_{\text{ent}} = \max_{\rho} I(X:R)_{\omega} \qquad r = \max_{\rho} H(X)_{\omega} - \max_{\rho} I(X:R)_{\omega}$$

- Hence, we determined the **information gain of quantum measurements**:

$$I(\mathcal{M}) = \max_{\rho} I(X:R)_{\omega}$$

- Result between classical [6] and quantum [7,8] reverse Shannon theorem
- Extension: rate region for non-feedback vs. feedback measurement simulation

$$c(r) = \max\left\{\max_{\rho} I(X:R)_{\omega}, \max_{\rho} H(X)_{\omega} - r\right\}$$

- Extension: quantum instrument simulation
- Extension: explicit protocols

[6] Bennett et al., IEEE Trans. on Inf. Th. (2002) [7] Bennett et al., arXiv:0912.5537v5 [8] Berta et al., Comm. in Math. Phys. (2011)

Thanks!