# Entanglement Cost of Quantum Channels

*Mario Berta*, *Fernando Brandao, Matthias Christandl, Stephanie Wehner*

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- *Application: an upper bound on the strong converse quantum capacity.*
- *Application: security in the noisy-storage model.*

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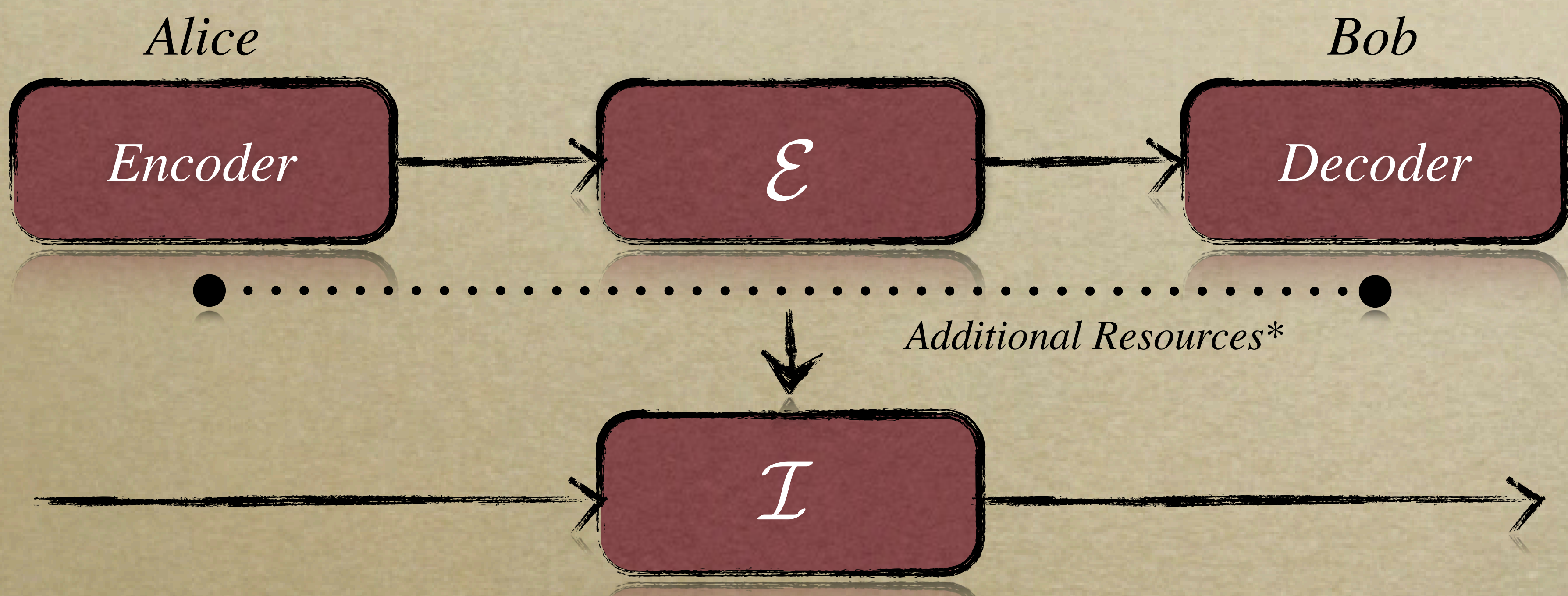


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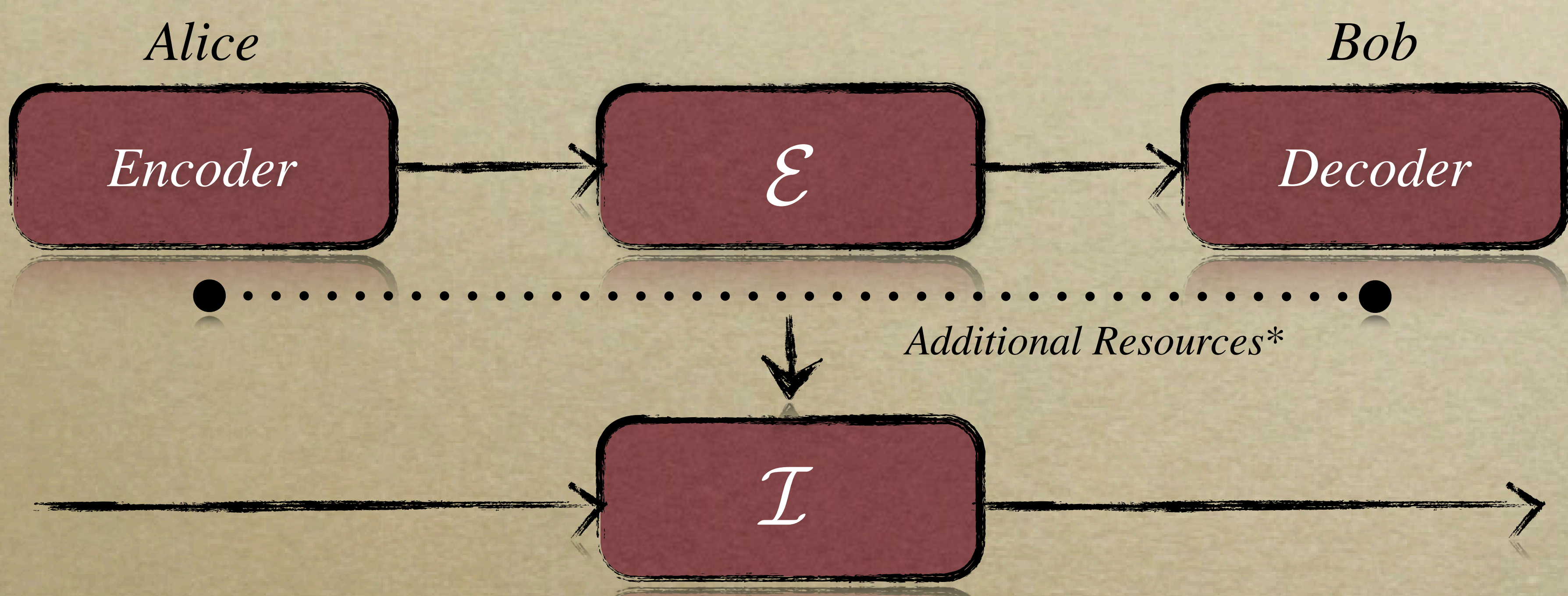


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$$Q, Q_E, Q_{\rightarrow}, Q_{\leftarrow}, Q_{\leftrightarrow}$$

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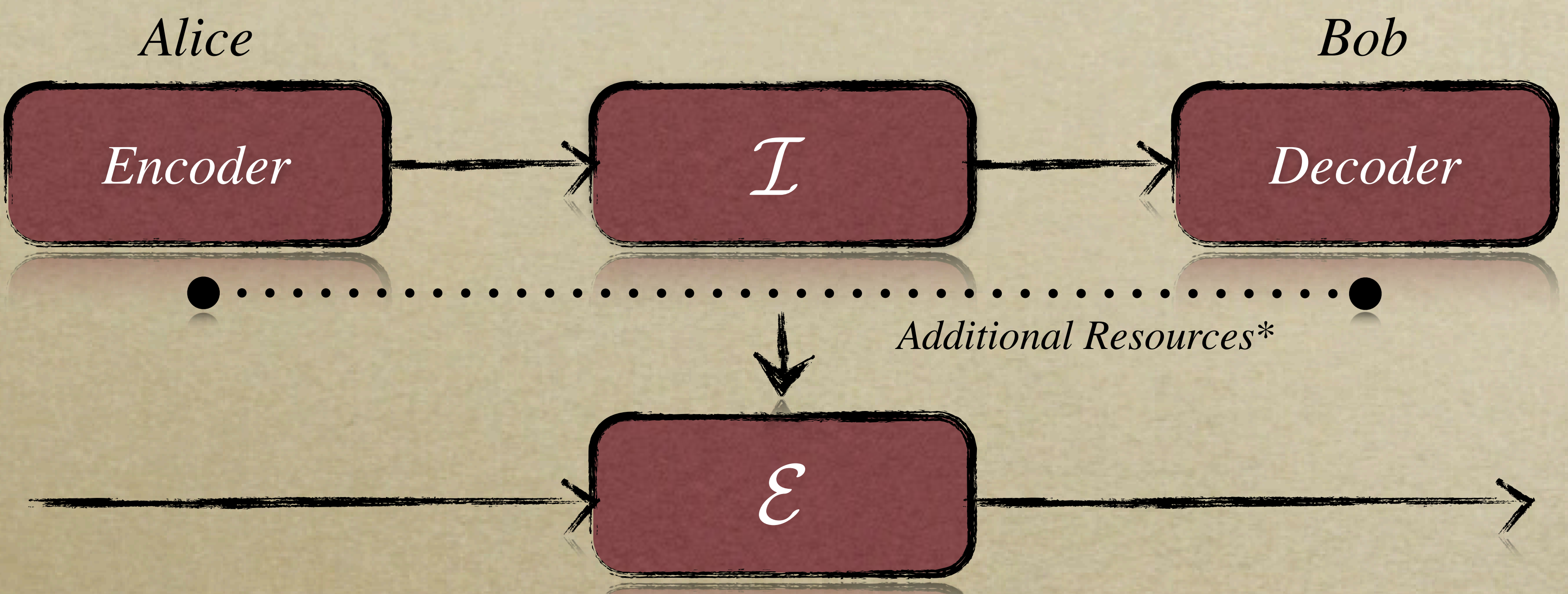


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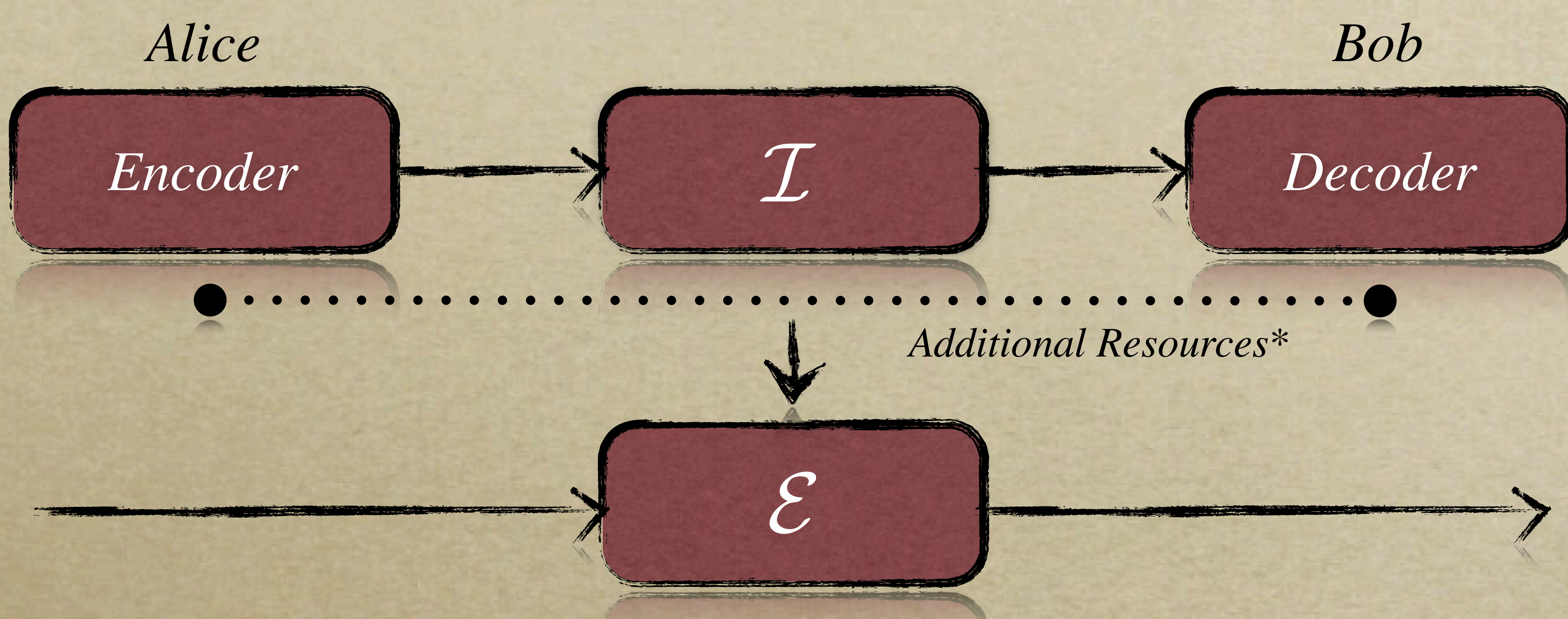


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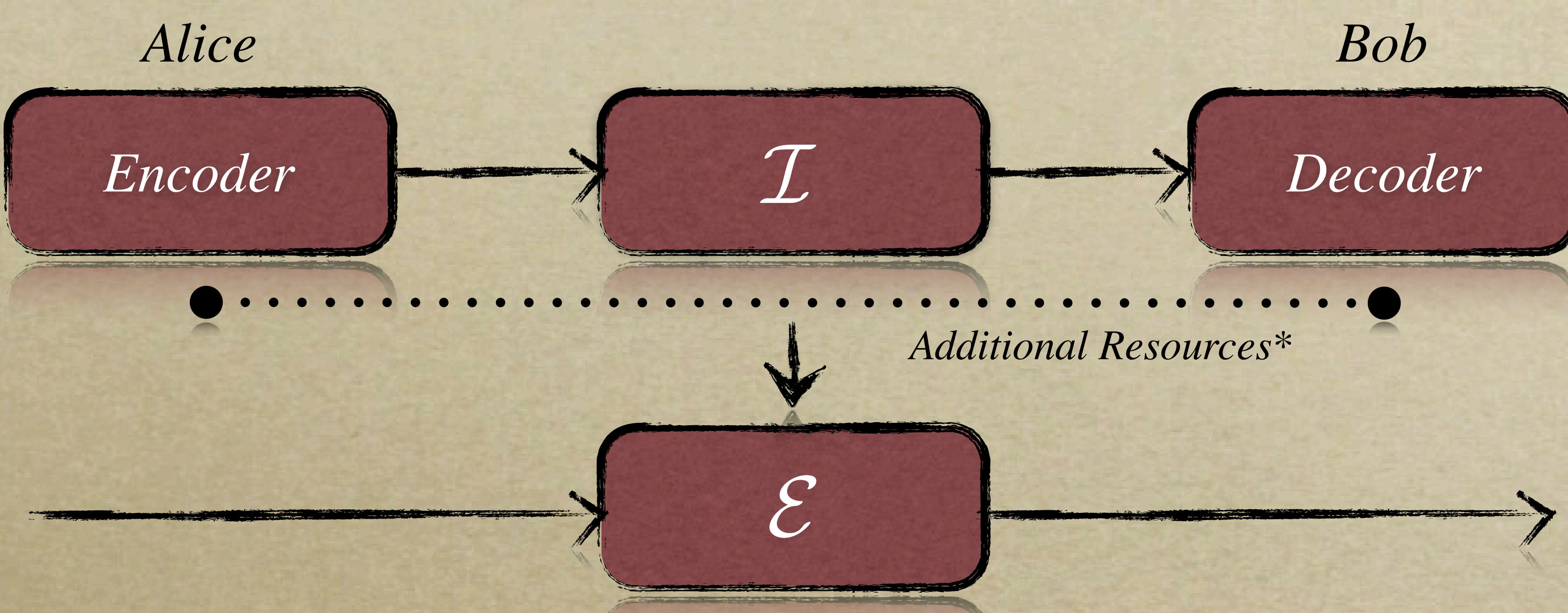


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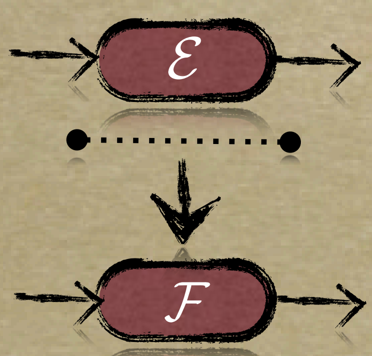
- *Asymptotic capacity of a quantum channel to simulate another quantum channel in the presence of free entanglement:*




$$Q_E(\mathcal{E},\mathcal{F}) = \frac{Q_E(\mathcal{E})}{Q_E(\mathcal{F})}$$

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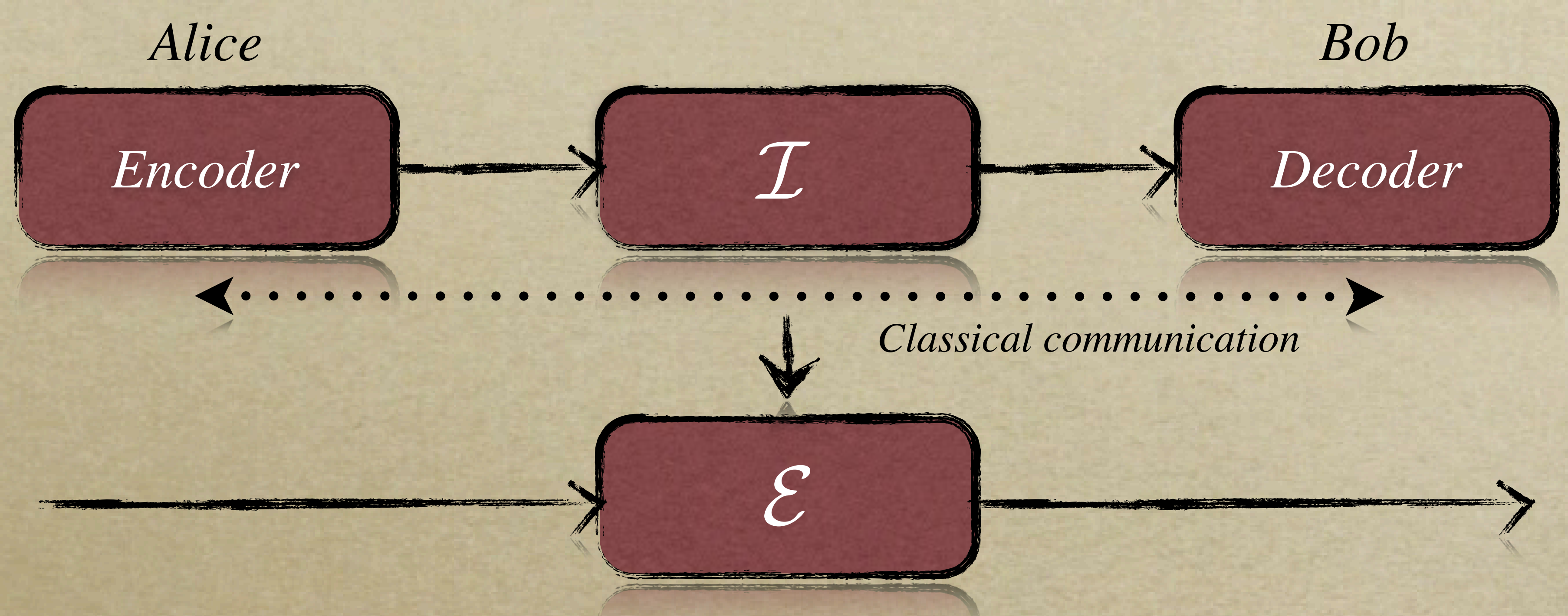


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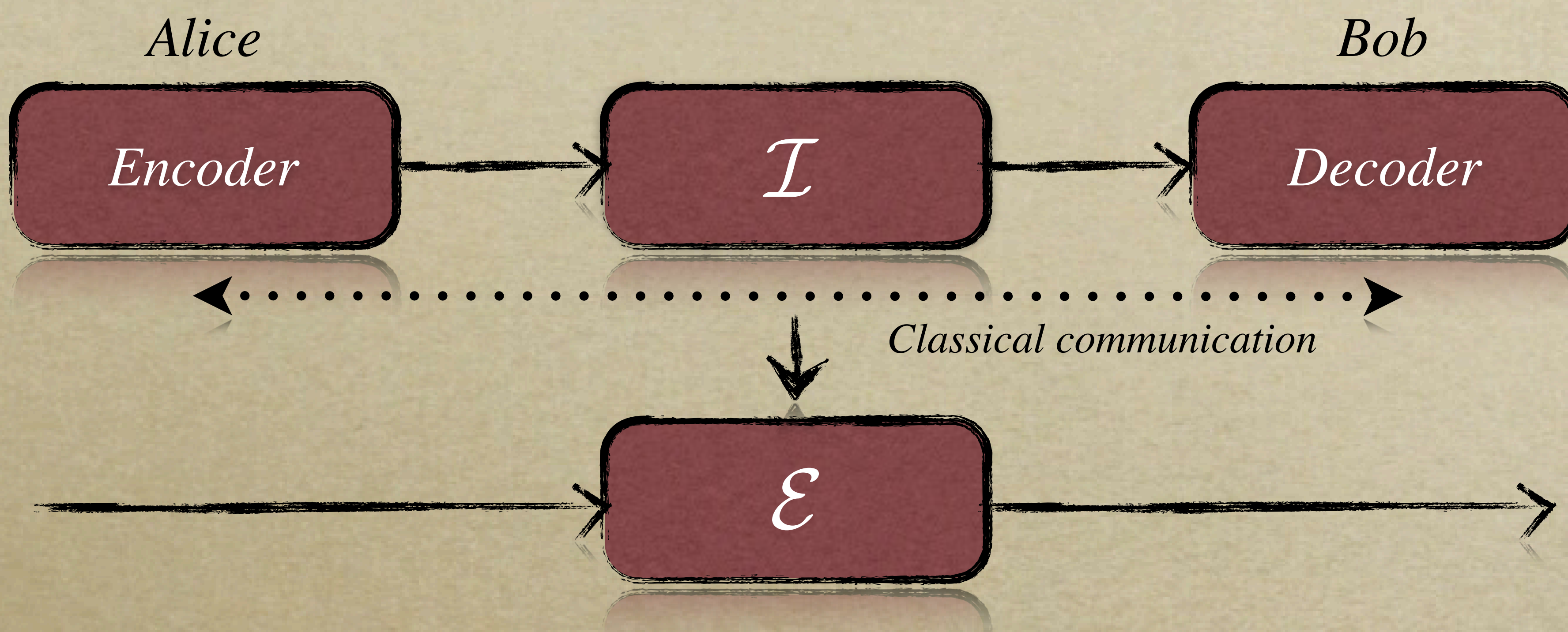


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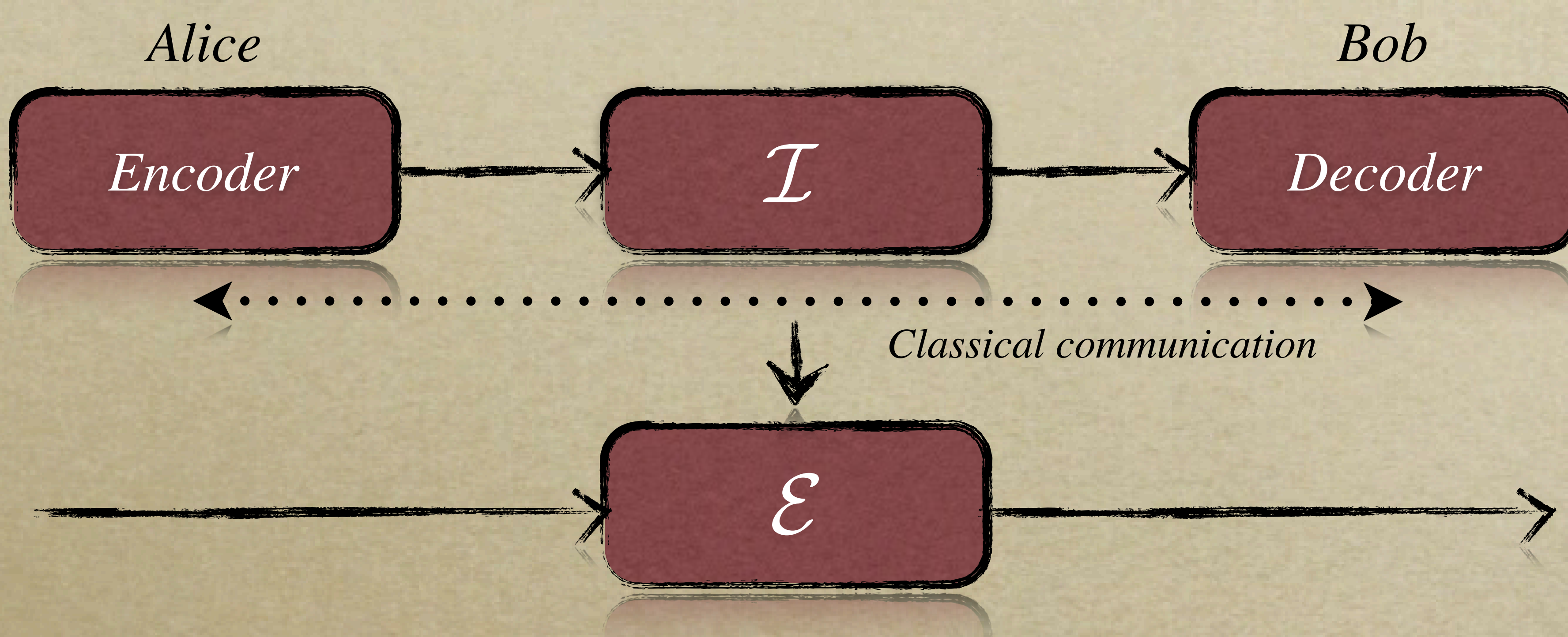


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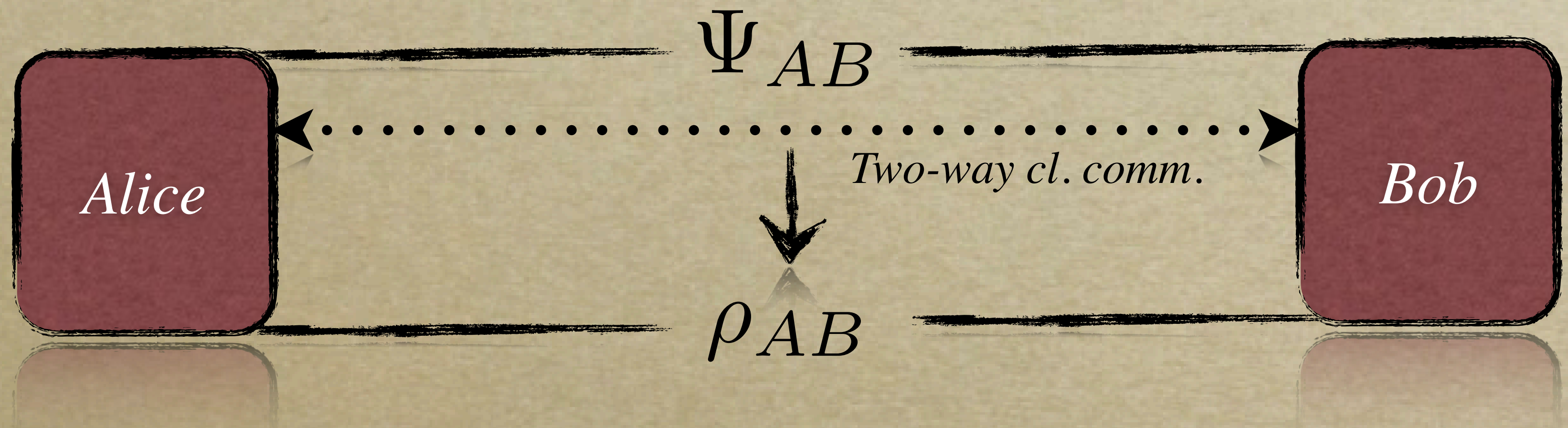



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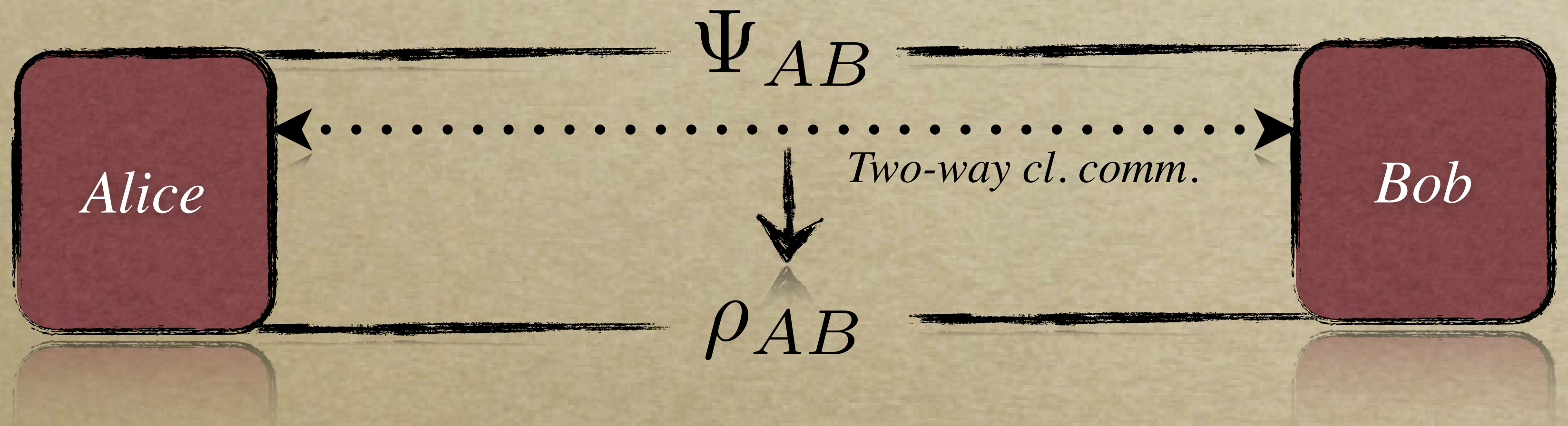


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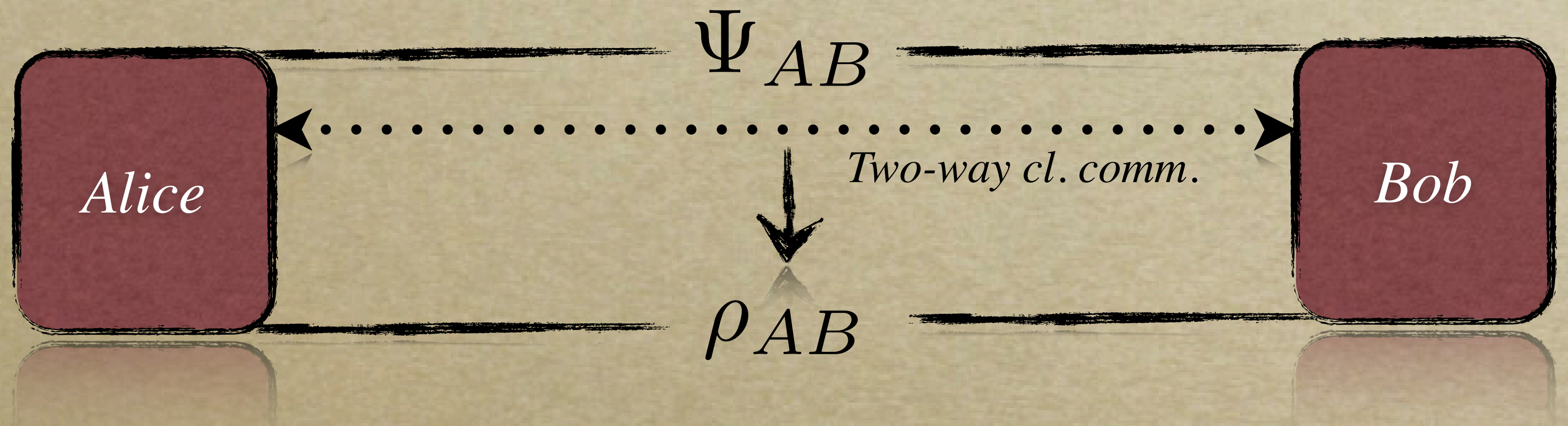


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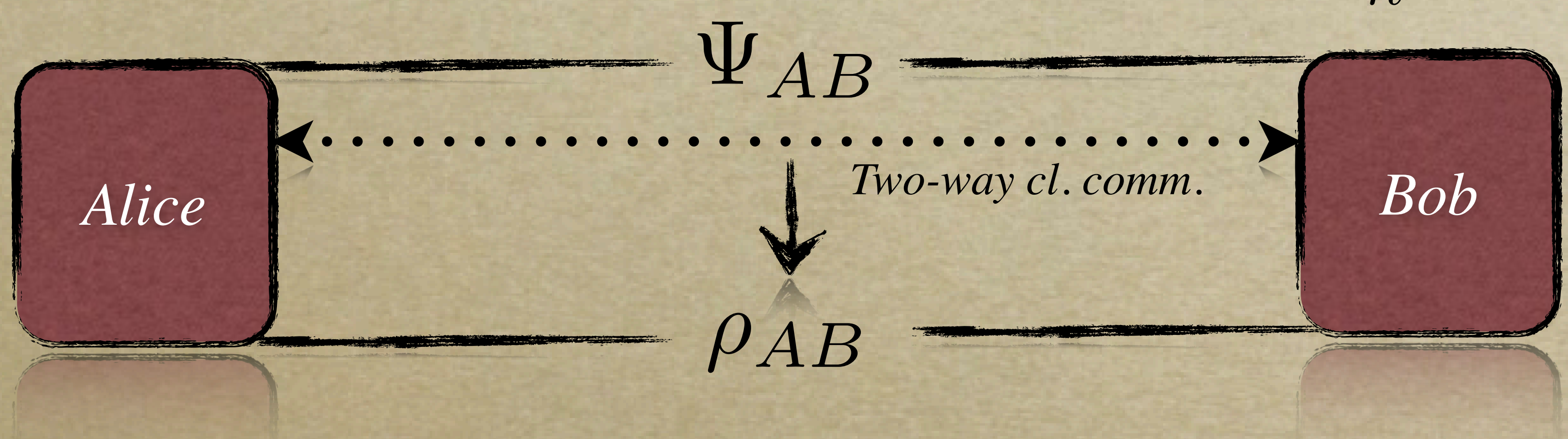


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# A few words about the achievability

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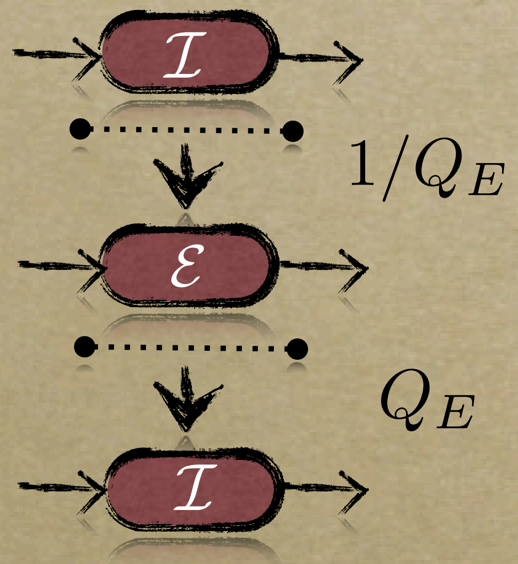



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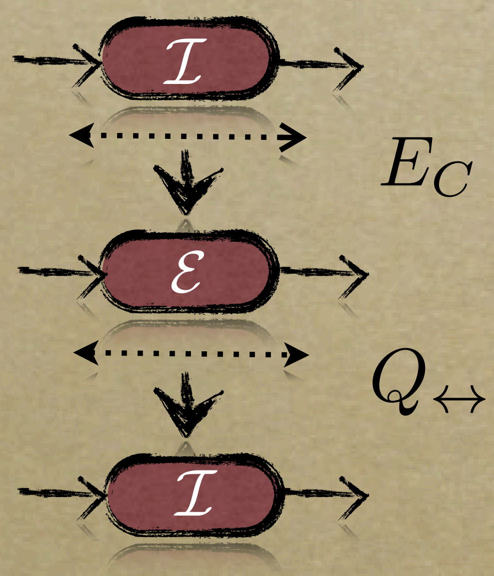


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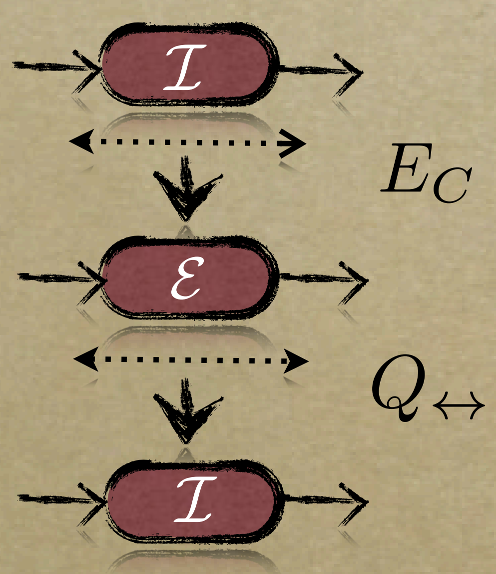


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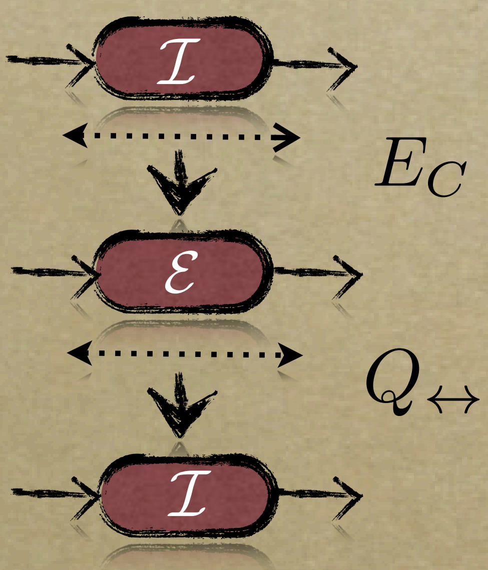


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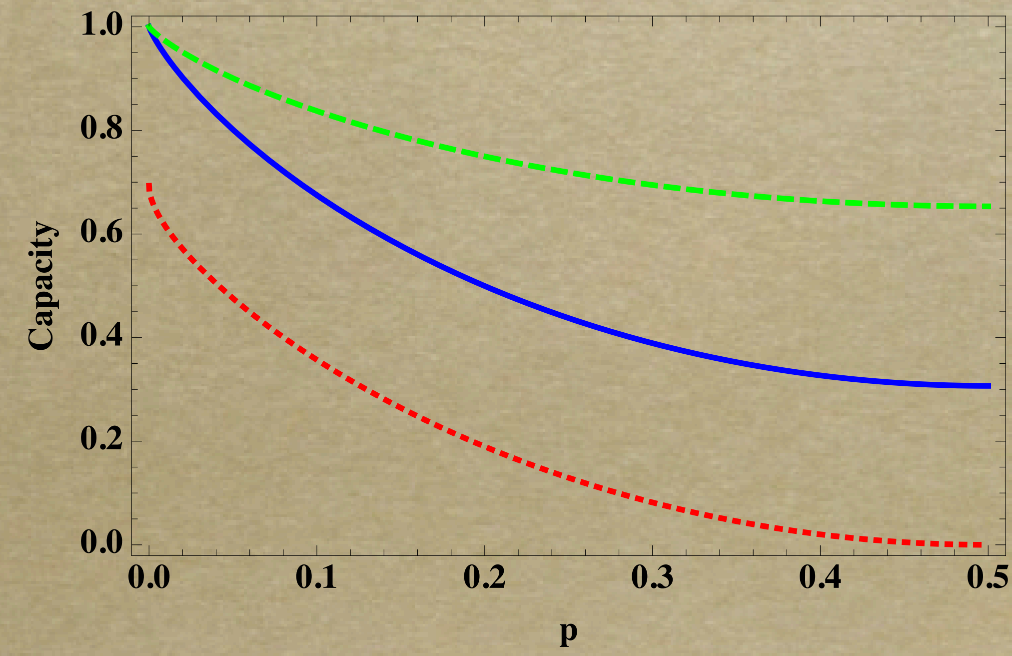
- *Qubit dephasing channel:*

$$\mathcal{E}(\rho) = (1-p) \cdot \rho + p \cdot \sigma_z \rho \sigma_z$$




*$Q = Q_{\rightarrow}$ (red)*
*vs. * (blue)*
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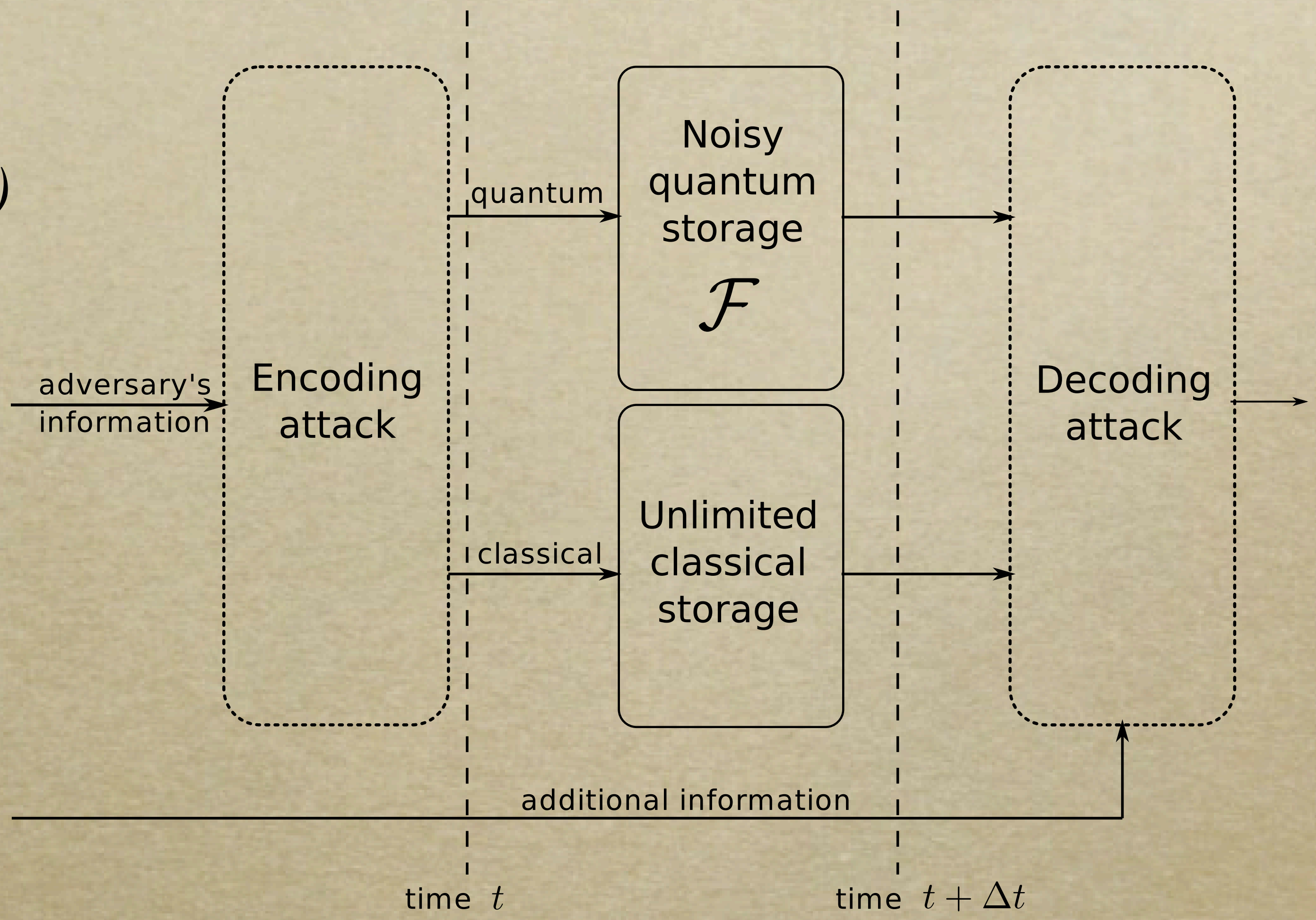



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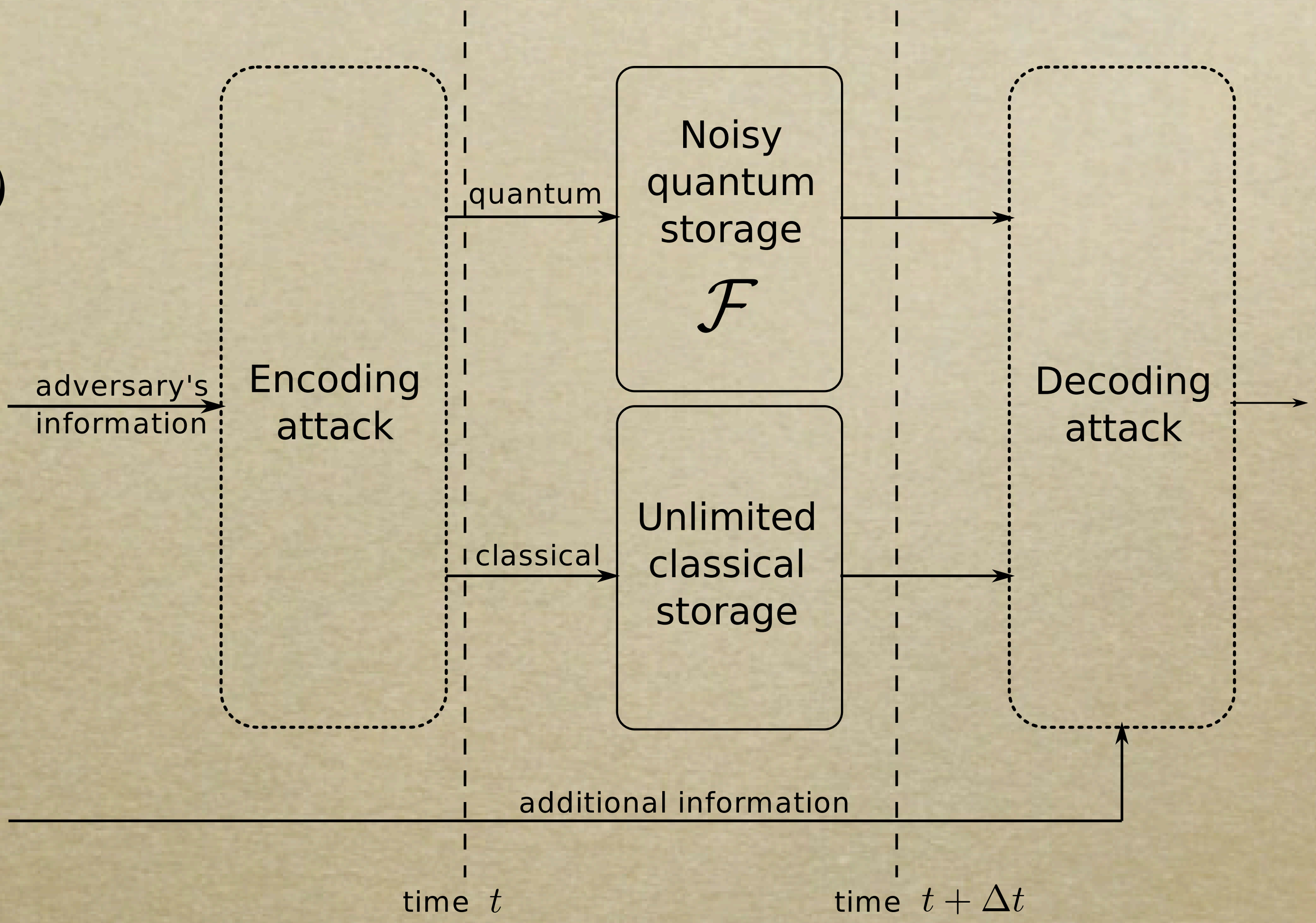


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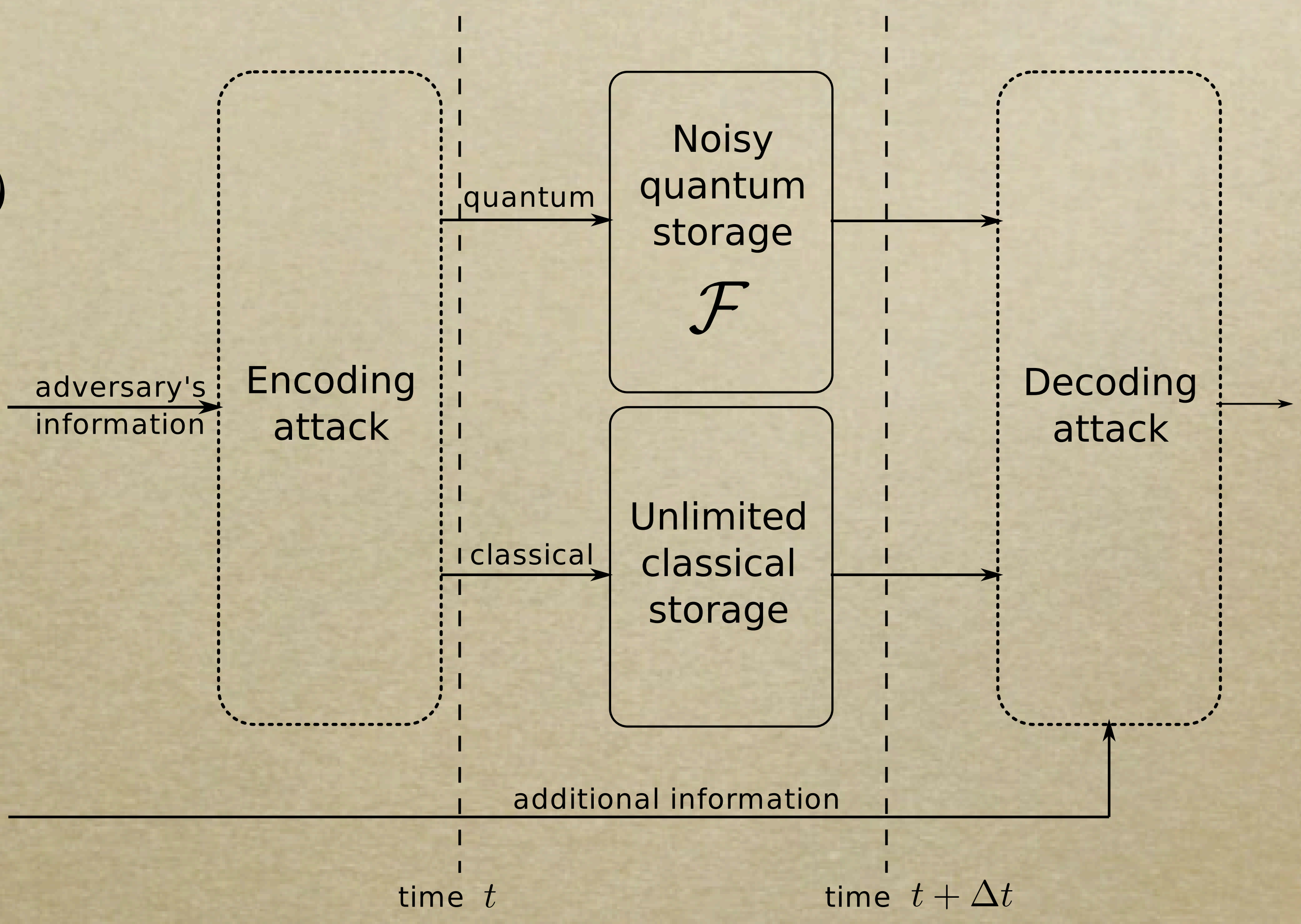



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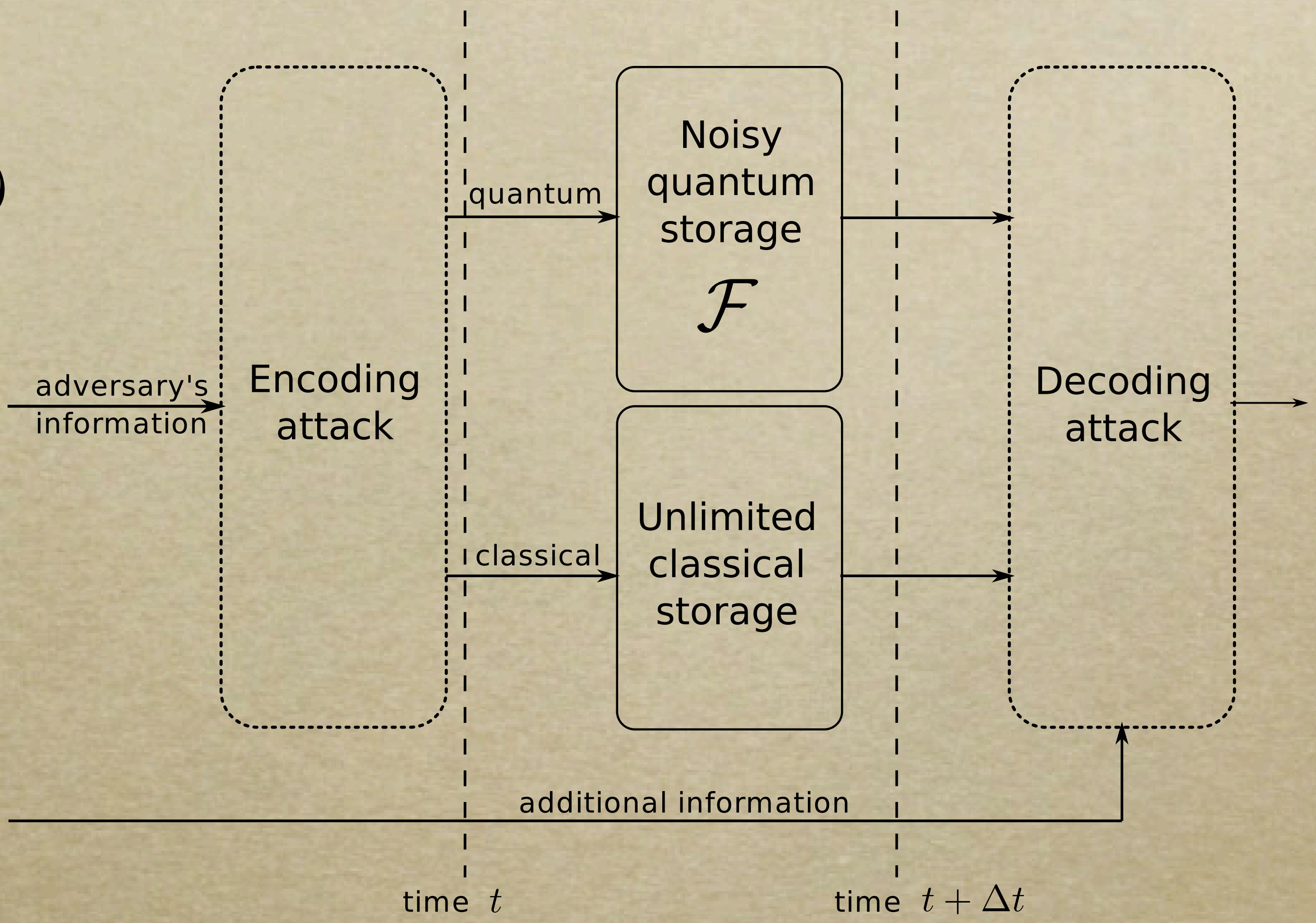


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- *New results in arXiv:1111.2026v3 (B., Fawzi, Wehner) --> ICITS 12, CRYPTO 12:*

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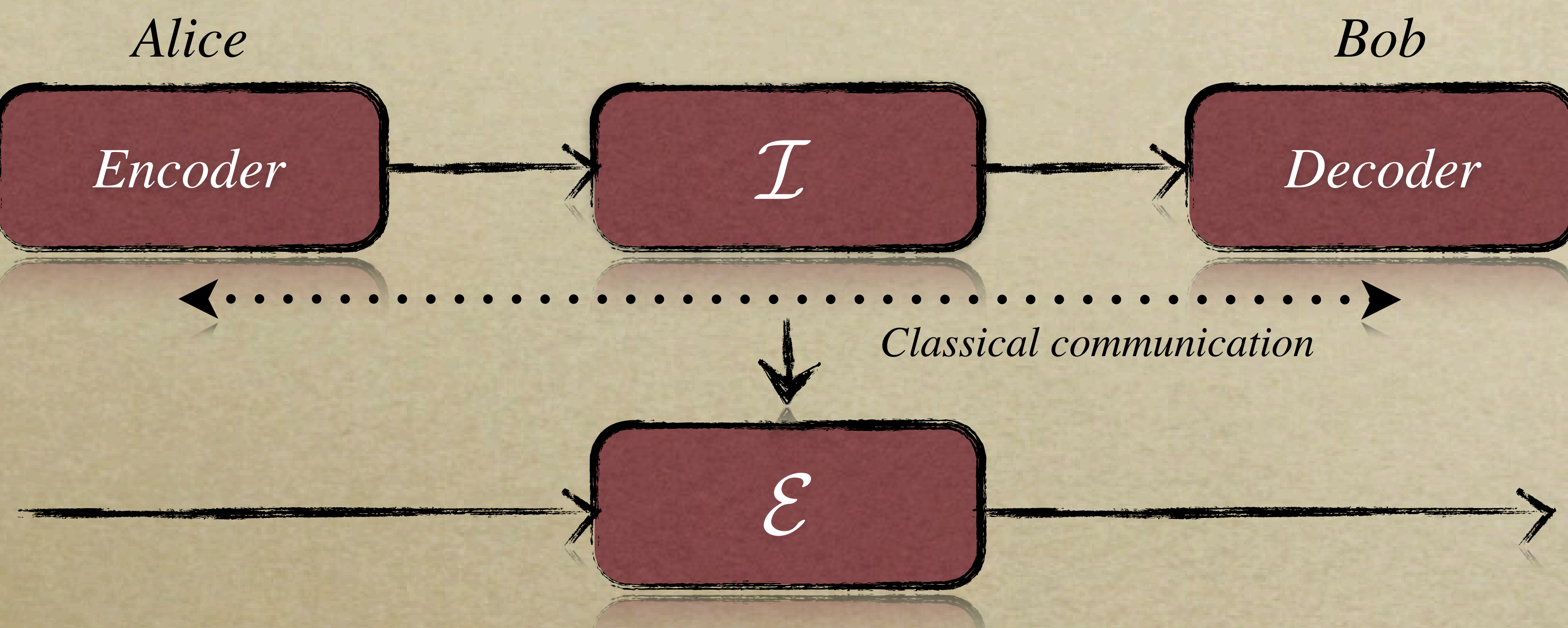
# Conclusions

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